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NONLINEAR DISPLACEMENT-BASED FINITE-ELEMENT ANALYSES OF COMPOSITE SHELLS—A NEW TOTAL LAGRANGIAN FORMULATION

PERNGJIN F. PAI and ANTHONY N. PALAZOTTO
Department of Aeronautics and Astronautics, Air Force Institute of Technology, Dayton, OH 45433, U.S.A.

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Abstract—Presented here is a new total Lagrangian finite-element formulation for general laminated composite shells undergoing large-displacement, large-rotation, and large-strain motion. The theory fully accounts for geometric nonlinearities (large rotations), large in-plane strains, general initial curvatures, transverse shear deformations, and interlaminar normal stresses by using Jaumann stress and strain measures, an exact coordinate transformation, and a new concept of orthogonal virtual rotations. In addition, with the inclusion of transverse normal and shear stresses, the theory accounts for three-dimensional stress effects and gives accurate effective structural stiffnesses. Because of the inclusion of all possible initial curvatures, the developed strain-displacement expressions can be used for any surface structures and are fully nonlinear, which contain von Karman strains as a special case. Moreover, the theory accounts for the continuity of interlaminar shear and peeling stresses and the normal and shear stress conditions on the bonding surfaces by using a new layer-wise local displacement field, which results in a higher-order shear theory that contains most shear deformation theories as special cases. In this paper, we develop a fully nonlinear displacement-based finite-element formulation based on the derived shell theory. The verification and accuracy of the theory will be presented in a subsequent paper by comparing obtained numerical results with some exact solutions.

1. INTRODUCTION

The advent and expanded use of high-performance composite materials in aircraft, aerospace vehicles, large space structures, automotive, shipbuilding, and recreational industries, have stimulated interest in the development of refined two-dimensional theories and efficient computational models for anisotropic plates and shells in order to have accurate predictions of the static and dynamic responses and failure characteristics. The inherent anisotropy of composite materials offers linear elastic couplings among bending, extension, torsion, and shearing motions, thereby making it possible to satisfy sophisticated design criteria, but it also makes nonclassical three-dimensional effects significant.

Loads on a shell are carried by a combination of bending and stretching actions because extension and bending motions are coupled due to an initially curved shape. Hence, the strength of a shell depends on its shape as well as its material. Moreover, owing to the initial curvatures, curvilinear coordinate systems are needed in the formulation of shells. These facts complicate the modeling and analysis of shells.

Motion of a shell is described by the deformation of its reference surface and the shear warpings and transverse stretching of its cross-sections with respect to the deformed reference surface. Different approximations of the deformation of the reference surface result in different geometrically nonlinear shell theories, and different approximations of the cross-section warpings result in different shear-deformation theories.

1.1. *Transverse shear and normal stresses*

Love (1944) established the foundations of the classical linear theory of thin shells by applying Kirchhoff's hypothesis to the cross-section deformation and hence neglecting transverse shear strains. Based on an order-of-magnitude analysis, Koiter (1960) pointed out that refinements of Love's first approximation theory of thin elastic shells are meaningless unless the effects of transverse shear and normal stresses are taken into account.

For isotropic plates and shells, shear effects are significant only if they vibrate at high frequencies and/or they are thick. For composite structures, shear effects are usually significant because the ratio of the in-plane Young's moduli E_α ($\alpha = 1,2$) to the transverse shear moduli $G_{\alpha 3}$ is between 20 and 50 compared with 2.5 to 3.0 in isotropic materials. Gulati and Essenburg (1967), and Noor and Burton (1990) showed that transverse shear deformations can be more significant in anisotropic plates and shells than in isotropic or even orthotropic structures of the same geometry.

In thin structures made of conventional isotropic structural materials, transverse normal strains and stresses are very small compared to their in-plane counterparts because transverse normal strains are mainly due to Poisson effect and transverse normal stresses are almost zero. However, for anisotropic shells, because of anisotropy, nonuniformity of Poisson's ratios through the shell thickness, and/or external pressure loads on the bonding surfaces (e.g. internal pressures of tires), interlaminar normal stresses and strain energy can be very high [e.g. see Noor *et al.* (1990)]. Moreover, some researchers (Whitney, 1971; Whitney and Sun, 1974; Sivakumaran and Chia, 1985; Doxsee and Springer, 1991) showed that for fiber-reinforced composite shells that undergo large rotations and/or are subjected to thermal loading, the effects of transverse normal expansion may be quite significant. In the case of thermal loading, fiber-reinforced composite materials usually exhibit high thermal expansion coefficients in directions normal to the fibers and very small coefficients in the direction of the fibers. Since there are no fibers in the transverse direction of a laminated composite structure, significant expansion or contraction in this direction can occur when the structure is subjected to a change in temperature. When thermal and mechanical loads are combined to cause nonlinear deformations, the effects of transverse normal strains and stresses are certainly significant, and cannot be neglected.

Experimental results show that (a) the classical shell theory underpredicts the deflections and overpredicts the natural frequencies and critical buckling loads because the transverse shear strains are neglected, and (b) the effect of transverse shear deformations increases with increasing mode number. Moreover, transverse shear and normal stresses are sources promoting structural failures (such as delaminations) because laminated composites are weak in interlaminar shear and peeling strengths.

In linear structural analyses, although an indirect post-processing technique by using equilibrium equations can give reasonable solutions for interlaminar stresses, there is a concern for using this technique in general application. Moreover, this technique cannot be directly extended to analyse structures undergoing large rotations because nonlinear equilibrium equations need to be solved. Hence, to better predict structural dynamic responses and/or failure of laminated shells, transverse normal and shear deformations need to be directly included in the modeling.

When transverse shear and normal stresses are included, the stress state becomes three-dimensional. Thus, because of the characteristics of a three-dimensional stress state and anisotropy and heterogeneity of composite materials, general composite shells need to be treated as three-dimensional solids. A three-dimensional finite-element method may be the only way to analyse them. However, a three-dimensional finite-element analysis is very expensive, especially if certain accuracy is desired. Moreover, the use of three-dimensional elements would involve many degrees of freedom that may be unnecessary. Furthermore, since a shell is characterized by a smaller dimension in the thickness direction compared to the in-plane dimensions, numerical ill-conditioning may occur in a three-dimensional finite-element analysis.

Fortunately, the inertial forces due to transverse normal and shear displacements are negligible because they are relative displacements with respect to the deformed reference surface and are therefore much smaller than the global displacements of the reference surface, especially for shells undergoing large rigid-body movements as well as elastic deformations. In general, the effect of rotary inertias is small compared to the effect of shear deformations (Ambartsumyan, 1969; Wu and Vinson, 1969). This is especially true for shells undergoing lower-frequency vibrations. However, transverse normal and shear displacements represent extra degrees of freedom for the deformation of a shell and hence significantly affect its effective structural stiffnesses. Hence, for a two-dimensional shell theory, transverse normal and shear displacements need to be modeled in order to account for their influence on structural stiffnesses.

For thin shells, lamination theories that employ a single expansion for the displacement field throughout the entire laminated thickness are commonly used in dynamic analysis. There are several shear-deformation theories, such as the first-order, third-order (Bhimaraddi, 1984; Reddy and Liu, 1985; Dennis and Palazotto, 1989), and other higher-order theories (Mirsky and Herrmann, 1957; Zukas and Vinson, 1971; Whitney and Sun, 1974; Voyiadjis and Shi, 1991). The first-order shear theory and some higher-order shear theories need shear correction factors which depend on variations of the shear angles, complex stress states, and the shape of the shell, and cannot be easily determined. On the other hand, the third-order shear theory does not involve determination of any unknown shear coefficients. However, the third-order theory does not account for the continuity of interlaminar shear stresses, the elastic coupling of the two transverse shear deformations, and the influence of initial curvatures, and hence it is appropriate only for isotropic and one-layer orthotropic plates (Pai *et al.*, 1993; Pai and Nayfeh, 1994a).

For moderately thick composite shells, Di Sciuva (1987) used a piecewise linear displacement field to fulfill the interlaminar continuity conditions for shear stresses, and the resulting shear warping functions are structure-dependent. Unfortunately, this theory does not satisfy the free shear-stress conditions on the boundary surfaces. Pai *et al.* (1993) and Pai and Nayfeh (1994a) extended the piecewise linear displacement field used by Di Sciuva (1987) by using quadratic and cubic interpolation functions. This displacement field satisfies continuity conditions of interlaminar shear stresses, accommodates free shear-stress conditions on the bonding surfaces, and accounts for initial curvatures and non-uniform distributions of transverse shear stresses in each layer. This theory contains most shear theories as special cases and reveals the shear coupling effect. Transverse normal strains are commonly neglected in shear-deformation theories.

1.2. *Objective stress and strain measures*

In structural analyses, stress and strain measures need to be work-conjugate, objective, geometric, and directional in order to use, in the constitutive equation, the material constants obtained from experiments in which rigid-body rotations are prevented and engineering stress and strain measures are used (Malvern, 1969; Pai and Palazotto, 1994a). Objective stresses and strains are invariant under rigid-body motions and do not arise due to pure rigid-body rotations. For nonobjective strain measures, there are no material stiffness matrices that can relate them to their work-conjugate stress measures in the constitutive equations when large rotations are involved. Unfortunately, most strain measures are not objective except Green–Lagrange strains and Jaumann strains.

Total Lagrangian finite-element formulations by using Green–Lagrange strains and second Piola–Kirchhoff stresses have been well developed [e.g. Palazotto and Dennis (1992)]. However, there are several drawbacks in the use of Green–Lagrange strains and second Piola–Kirchhoff stresses. First, because Green–Lagrange strains are defined using the change in the squared length of an infinitesimal element, they are energy measures and not geometric measures. Second, because second Piola–Kirchhoff stresses are not defined as force per unit area and not along three perpendicular directions (Washizu, 1982), it is difficult to match them with stress conditions on the bonding surfaces in order to determine shear warping functions. Third, because Green–Lagrange strains and second Piola–Kirchhoff stresses are energy measures, their constitutive equation cannot use the material constants obtained from experiments using engineering stress and strain measures, which are geometric measures (Pai and Nayfeh, 1994b). Moreover, in order to perform stress analysis and interlaminar failure analysis of a geometrically nonlinear structure, second Piola–Kirchhoff stresses are usually transformed into Cauchy stresses, which are geometric measures. However, the computation of such a transformation is costly (Norwood *et al.*, 1991). Even if Cauchy stresses are obtained, it is still difficult to match Cauchy stresses with the real boundary conditions or to use them in the analyses of laminated composites because the directions of Cauchy stresses are defined with respect to the undeformed system configuration and do not act truly normal or parallel to the deformed reference surface of the structure. These drawbacks can be avoided by using Jaumann stress and strain measures.

Jaumann (or Jaumann–Biot–Cauchy) strains are defined by using the right stretch tensor, which is obtained from the deformation gradient by using the polar decomposition theory (Malvern, 1969). These strains were shown by Fraeijs de Veubeke (1972) to be objective engineering strain measures. However, Fraeijs de Veubeke (1972) did not discuss the directions of Jaumann strains and their relations to other strain measures, which are important in large-deformation analyses of composite structures because the stiffnesses of composite materials are direction-dependent. Pai and Nayfeh (1994b), and Pai and Palazotto (1994a) showed that Jaumann strains are objective engineering strains and have directions along the deformed system configuration. Hence, experimentally obtained material constants can be used in the constitutive equation of Jaumann stresses and strains. Because Jaumann strains and stresses are objective even when large rotations are involved, they can be used to model structures undergoing large rotations and strains.

In analyzing nonlinear structural problems, either the total Lagrangian (T.L.) formulation or the updated Lagrangian (U.L.) is used. In the T.L. formulation, all displacements are referred to the initial configuration at time $t = 0$ and fully nonlinear equations are used. This method requires a nonlinear algorithm to solve the equations of motion, where an iterative procedure is usually required. In a U.L. formulation, all displacements are referred to the coordinates at the end of the previous load step. Both the T.L. and U.L. formulations include all kinematic nonlinear effects due to large displacements, large rotations and large strains. However, in the U.L. formulation, displacements and stresses need to be transformed and updated. Moreover, because the updated deformed volume of the previous step is used in the current step of computation, there is always an approximation involved and such errors can be accumulated. Hence, the T.L. formulation is chosen in this paper. However, to use the T.L. formulation, fully nonlinear equations of motion need to be derived.

1.3. *Large rotations*

There are some difficulties in the development of fully nonlinear composite plate and shell models accounting for large rotations. The fundamental work on geometrically nonlinear theories of plates and shells is due to von Karman (1910). The von Karman theory is based on the hypothesis that only the derivatives of transverse deflection are significant in the nonlinear strain-displacement equations. Although von Karman strains produce some quadratic and cubic terms in the equations of motion, Pai and Nayfeh (1991) and Sacco and Reddy (1992) showed that von Karman strains do not account for all nonlinearities due to large rotations and hence cannot be used to derive a fully nonlinear two-dimensional structural theory.

To model large rotations of a structure, some researchers used Euler-like angles. However, since finite rotations are sequence-dependent quantities, their expressions are not unique. Moreover, the variations of Euler-like angles are not along three orthogonal directions. Pai and Nayfeh (1991, 1992) used Jaumann stress and strain measures and a local reference frame, and introduced a new interpretation and manipulation of orthogonal virtual rotations in deriving geometrically exact plate and beam theories, where the large structural rotations are fully described by the transformation matrix which relates the local reference frame and the undeformed system frame. The use of these new concepts makes it possible to fully correlate Newtonian and energy approaches in nonlinear formulations. The current numerical results obtained by Pai and Palazotto (1994b) show the merit of using Jaumann stress and strain measures and these new concepts in the modeling and analysis of flexible structures undergoing very large deformations and rotations.

1.4. *Unification of plate and shell theories*

Pai and Nayfeh (1994a) pointed out that both plates and shells are two-dimensional structures described by two orthogonal curvilinear (or rectilinear if rectangular plates) coordinates and one rectilinear coordinate perpendicular to the reference surface, and hence the only difference in the modeling of plates and shells is the initial curvatures of the two coordinates on the reference surface. Hence, the formulations of plates and shells can be simplified by using initial curvatures instead of Lamé parameters and can be unified into one with all possible curvatures included.

Here, we present a unified geometrically exact theory for laminated plates and shells. The theory fully accounts for geometric nonlinearities (large rotations and displacements) and extensionality by using Jaumann stresses and strains and an exact coordinate transformation, which result in exact nonlinear curvature and strain-displacement expressions. Transverse shear deformations and peeling stresses are accounted for by using a layer-wise higher-order displacement field, which accounts for the continuity of interlaminar shear and normal stresses and does not require shear correction factors. Based on the developed shell theory, discretized incremental finite-element equations are derived using the extended Hamilton principle.

2. SURFACE ANALYSIS

2.1. *Curvatures, in-plane strains, and coordinate systems*

Deformations of the reference surface of a shell can be described by two coordinate systems: one describes the undeformed reference surface and the other describes the deformed reference surface. We consider a toroidal shell having thickness h and two constant radii of curvatures r and R over the domain $0 \leqslant \alpha \leqslant 2\pi$ and $\Theta_1 \leqslant \theta \leqslant \Theta_2$, as shown in Fig. 1(a). The frame xyz is an orthogonal curvilinear coordinate system with the curvilinear axes x and y being on the undeformed reference surface and the z axis being a rectilinear axis, and the frame $\xi\eta\zeta$ is an orthogonal curvilinear coordinate system with the curvilinear axes ξ and η being on the deformed reference surface and the ζ axis being a rectilinear axis. The frame abc is a body-fixed rectangular coordinate system, which is used, for reference purposes, in the calculation of the initial curvatures and inertial forces due to rigid-body motions. We let $\mathbf{j}_1$, $\mathbf{j}_2$, and $\mathbf{j}_3$ denote the unit vectors along the axes x, y, and z; $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ denote the unit vectors along the axes ξ, η, and ζ; and $\mathbf{i}_a$, $\mathbf{i}_b$, and $\mathbf{i}_c$ denote the unit vectors along the axes a, b, and c. It follows from Fig. 1(a) that

$$\mathrm{d}x = (R - r\sin\theta)\,\mathrm{d}\alpha, \quad \mathrm{d}y = r\,\mathrm{d}\theta \tag{1}$$

and the position vector $\mathbf{P}$ of the observed point A is given by

$$\mathbf{P} = (R - r\sin\theta)\sin\alpha\,\mathbf{i}_a + (R - r\sin\theta)\cos\alpha\,\mathbf{i}_b - r\cos\theta\,\mathbf{i}_c. \tag{2}$$

Taking the first- and second-order derivatives of eqn (2) with respect to x and y yields

$$\mathbf{j}_1 = \frac{\partial \mathbf{P}}{\partial x} = \cos\alpha\,\mathbf{i}_a - \sin\alpha\,\mathbf{i}_b$$

$$\mathbf{j}_2 = \frac{\partial \mathbf{P}}{\partial y} = -\cos\theta\sin\alpha\,\mathbf{i}_a - \cos\theta\cos\alpha\,\mathbf{i}_b + \sin\theta\,\mathbf{i}_c$$

$$\mathbf{j}_3 = \mathbf{j}_1 \times \mathbf{j}_2 = -\sin\theta\sin\alpha\,\mathbf{i}_a - \sin\theta\cos\alpha\,\mathbf{i}_b - \cos\theta\,\mathbf{i}_c$$

and

$$\frac{\partial}{\partial x}\{\mathbf{j}_{123}\} = [K_1^0]\{\mathbf{j}_{123}\}, \quad [K_1^0] \equiv \begin{bmatrix} 0 & k_5^0 & -k_1^0 \\ -k_5^0 & 0 & -k_{61}^0 \\ k_1^0 & k_{61}^0 & 0 \end{bmatrix} \tag{3a}$$

$$\frac{\partial}{\partial y}\{\mathbf{j}_{123}\} = [K_2^0]\{\mathbf{j}_{123}\}, \quad [K_2^0] \equiv \begin{bmatrix} 0 & k_4^0 & -k_{62}^0 \\ -k_4^0 & 0 & -k_2^0 \\ k_{62}^0 & k_2^0 & 0 \end{bmatrix}, \tag{3b}$$

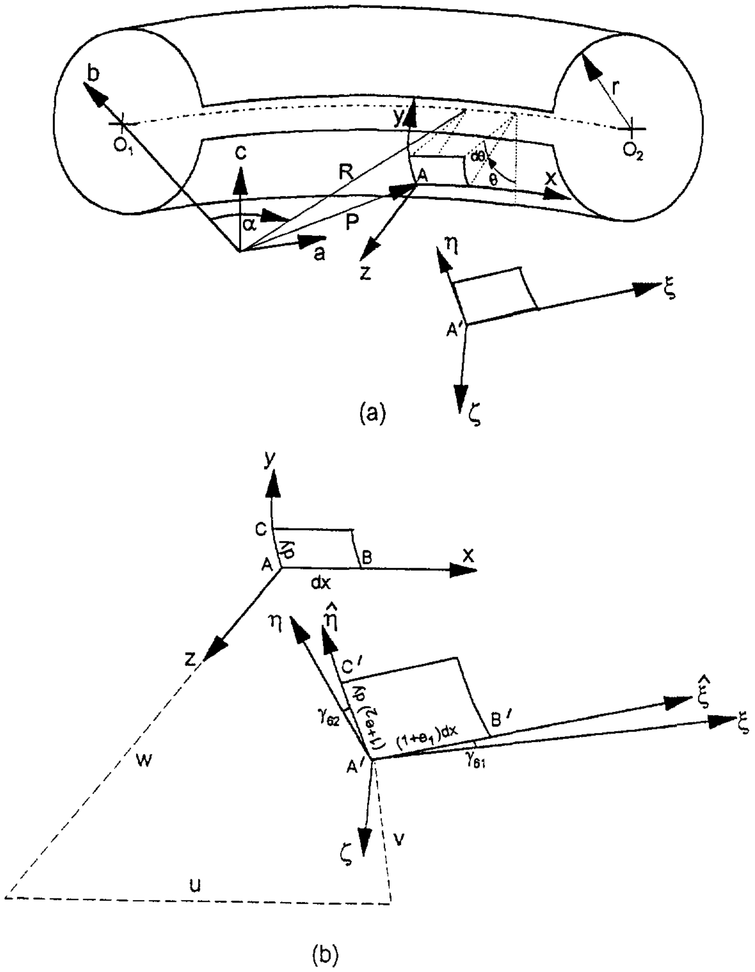

Fig. 1. The deformation of a shell element: (a) xyz = orthogonal curvilinear frame, where the x–y surface represents the undeformed reference surface, and $\xi\eta\zeta$ = orthogonal curvilinear frame, where the ξ–η surface represents the deformed reference surface; and (b) the undeformed and deformed geometries of an element of the reference surface.

where $\{\mathbf{j}_{123}\} \equiv \{\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3\}^T$, $[K_1^0]$ and $[K_2^0]$ are the initial curvature matrices, and the initial curvatures are given by

$$k_1^0 \equiv \frac{\partial \mathbf{j}_3}{\partial x}\cdot\mathbf{j}_1 = -\frac{\partial \mathbf{j}_1}{\partial x}\cdot\mathbf{j}_3 = \frac{-\sin\theta}{R-r\sin\theta}, \quad k_{61}^0 = k_{62}^0 = k_4^0 = 0,$$

$$k_2^0 = \frac{1}{r}, \quad k_5^0 = \frac{\cos\theta}{R-r\sin\theta}. \tag{4}$$

For circular cylindrical shells, $R \to \infty$ and hence $k_1^0 = k_5^0 = 0$. For general shells, all the initial curvatures in eqns (3) are nontrivial (Pai and Nayfeh, 1994a), and hence we will keep all initial curvatures in the following derivations, instead of using eqn (4).

Figure 1(b) shows an element of the reference surface before and after deformation. Here, u, v, and w are the displacements of the observed reference point A with respect to the axes x, y, and z, respectively. The axes $\hat{\xi}$ and $\hat{\eta}$ represent the deformed configurations of the axes x and y, respectively; and γ_6 $(= \gamma_{61}+\gamma_{62})$ is the in-plane shear deformation. We let $\mathbf{i}_{\hat{1}}$ and $\mathbf{i}_{\hat{2}}$ denote the unit vectors along the axes $\hat{\xi}$ and $\hat{\eta}$, respectively. We note that the axes ξ and η coincide with the axes $\hat{\xi}$ and $\hat{\eta}$ only if the in-plane shear deformation γ_6 is zero.

Letting $\mathbf{D} \equiv u\mathbf{j}_1 + v\mathbf{j}_2 + w\mathbf{j}_3$ and using eqns (3), we obtain

$$\overrightarrow{A'B'} = \mathrm{d}x\mathbf{j}_1 + \frac{\partial \mathbf{D}}{\partial x}\mathrm{d}x$$
$$= [(1+u_x - vk_5^0 + wk_1^0)\mathbf{j}_1 + (v_x + uk_5^0 + wk_{61}^0)\mathbf{j}_2 + (w_x - uk_1^0 - vk_{61}^0)\mathbf{j}_3]\,\mathrm{d}x$$
$$\overrightarrow{A'C'} = \mathrm{d}y\mathbf{j}_2 + \frac{\partial \mathbf{D}}{\partial y}\mathrm{d}y$$
$$= [(u_y - vk_4^0 + wk_{62}^0)\mathbf{j}_1 + (1+v_y + uk_4^0 + wk_2^0)\mathbf{j}_2 + (w_y - uk_{62}^0 - vk_2^0)\mathbf{j}_3]\,\mathrm{d}y, \tag{5}$$

where $(\ )_x \equiv \partial(\ )/\partial x$ and $(\ )_y \equiv \partial(\ )/\partial y$. Hence, the axial strains along the axes $\hat{\xi}$ and $\hat{\eta}$ are e_1 and e_2 and are given by

$$e_1 = \frac{\overline{A'B'} - \mathrm{d}x}{\mathrm{d}x}$$
$$= \sqrt{(1+u_x - vk_5^0 + wk_1^0)^2 + (v_x + uk_5^0 + wk_{61}^0)^2 + (w_x - uk_1^0 - vk_{61}^0)^2} - 1 \tag{6}$$
$$e_2 = \frac{\overline{A'C'} - \mathrm{d}y}{\mathrm{d}y}$$
$$= \sqrt{(uy - vk_4^0 + wk_{62}^0)^2 + (1+v_y + uk_4^0 + wk_2^0)^2 + (w_y - uk_{62}^0 - vk_2^0)^2} - 1. \tag{7}$$

The unit vectors along the $\hat{\xi}$ and $\hat{\eta}$ directions are given by

$$\mathbf{i}_1' = \frac{\overrightarrow{A'B'}}{(1+e_1)\,\mathrm{d}x} = \hat{T}_{11}\mathbf{j}_1 + \hat{T}_{12}\mathbf{j}_2 + \hat{T}_{13}\mathbf{j}_3 \tag{8}$$

$$\mathbf{i}_2' = \frac{\overrightarrow{A'C'}}{(1+e_2)\,\mathrm{d}y} = \hat{T}_{21}\mathbf{j}_1 + \hat{T}_{22}\mathbf{j}_2 + \hat{T}_{23}\mathbf{j}_3, \tag{9}$$

where

$$\hat{T}_{11} = \frac{1+u_x - vk_5^0 + wk_1^0}{1+e_1}, \quad \hat{T}_{12} = \frac{v_x + uk_5^0 + wk_{61}^0}{1+e_1}, \quad \hat{T}_{13} = \frac{w_x - uk_1^0 - vk_{61}^0}{1+e_1} \tag{10}$$

$$\hat{T}_{21} = \frac{u_y - vk_4^0 + wk_{62}^0}{1+e_2}, \quad \hat{T}_{22} = \frac{1+v_y + uk_4^0 + wk_2^0}{1+e_2}, \quad \hat{T}_{23} = \frac{w_y - uk_{62}^0 - vk_2^0}{1+e_2}. \tag{11}$$

Using eqns (8) and (9), we obtain

$$\gamma_6 = \gamma_{61} + \gamma_{62} = \sin^{-1}(\mathbf{i}_1' \cdot \mathbf{i}_2') = \sin^{-1}(\hat{T}_{11}\hat{T}_{21} + \hat{T}_{12}\hat{T}_{22} + \hat{T}_{13}\hat{T}_{23}). \tag{12a}$$

Hence, γ_6 can be represented in terms of u, v, and w. However, to determine the unique expressions of γ_{61} and γ_{62}, we need to use the relation

$$(1+e_1)\sin\gamma_{61} = (1+e_2)\sin\gamma_{62}, \tag{12b}$$

which results from the fact that Jaumann strains are symmetric (Pai and Palazotto, 1994a).

The unit normal to the deformed reference plane is given by

$$\mathbf{i}_3 = \frac{\mathbf{i}_1' \times \mathbf{i}_2'}{|\mathbf{i}_1' \times \mathbf{i}_2'|} = T_{31}\mathbf{j}_1 + T_{32}\mathbf{j}_2 + T_{33}\mathbf{j}_3, \tag{13}$$

where

$$T_{31} = (\hat{T}_{12}\hat{T}_{23} - \hat{T}_{13}\hat{T}_{22})/R_0,\ T_{32} = (\hat{T}_{13}\hat{T}_{21} - \hat{T}_{11}\hat{T}_{23})/R_0\ T_{33} = (\hat{T}_{11}\hat{T}_{22} - \hat{T}_{12}\hat{T}_{21})/R_0,$$

$$R_0 \equiv \sqrt{(\hat{T}_{12}\hat{T}_{23} - \hat{T}_{13}\hat{T}_{22})^2 + (\hat{T}_{13}\hat{T}_{21} - \hat{T}_{11}\hat{T}_{23})^2 + (\hat{T}_{11}\hat{T}_{22} - \hat{T}_{12}\hat{T}_{21})^2} = |\cos\gamma_6|. \quad (14)$$

Using eqns (8), (9), and (13) and Fig. 1(b), we obtain the following transformation which relates the undeformed coordinate system xyz to the deformed coordinate system $\xi\eta\zeta$:

$$\{\mathbf{i}_{123}\} = [T]\{\mathbf{j}_{123}\},\quad [T] \equiv \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = [\Gamma]\begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} & \hat{T}_{13} \\ \hat{T}_{21} & \hat{T}_{22} & \hat{T}_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}, \quad (15a)$$

where

$$\{\mathbf{i}_{123}\} = [\Gamma]\{\mathbf{i}_{\hat{1}\hat{2}3}\}$$

$$[\Gamma] \equiv \begin{bmatrix} \cos\gamma_{61} & \sin\gamma_{61} & 0 \\ \sin\gamma_{62} & \cos\gamma_{62} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{\cos\gamma_6}\begin{bmatrix} \cos\gamma_{62} & -\sin\gamma_{61} & 0 \\ -\sin\gamma_{62} & \cos\gamma_{61} & 0 \\ 0 & 0 & \cos\gamma_6 \end{bmatrix}$$

$$\{\mathbf{i}_{123}\} \equiv \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}^{\mathrm{T}},\quad \text{and}\quad \{\mathbf{i}_{\hat{1}\hat{2}3}\} \equiv \{\mathbf{i}_{\hat{1}}, \mathbf{i}_{\hat{2}}, \mathbf{i}_3\}^{\mathrm{T}}. \quad (15b)$$

Using eqns (15a) and (3a, b) and the identities

$$\frac{\partial \mathbf{i}_j}{\partial x}\cdot\mathbf{i}_j = \frac{\partial \mathbf{i}_j}{\partial y}\cdot\mathbf{i}_j = 0,\quad \frac{\partial \mathbf{i}_j}{\partial x}\cdot\mathbf{i}_k = -\frac{\partial \mathbf{i}_k}{\partial x}\cdot\mathbf{i}_j,\quad \frac{\partial \mathbf{i}_j}{\partial y}\cdot\mathbf{i}_k = -\frac{\partial \mathbf{i}_k}{\partial y}\cdot\mathbf{i}_j \quad \text{for}\quad j,k = 1,2,3, \quad (16)$$

where repeated indices do not imply summations, we obtain

$$\frac{\partial}{\partial x}\{\mathbf{i}_{123}\} = [K_1]\{\mathbf{i}_{123}\}, [K_1] \equiv \begin{bmatrix} 0 & k_5 & -k_1 \\ -k_5 & 0 & -k_{61} \\ k_1 & k_{61} & 0 \end{bmatrix} = \frac{\partial[T]}{\partial x}[T]^{\mathrm{T}} + [T][K_1^0][T]^{\mathrm{T}} \quad (17)$$

$$\frac{\partial}{\partial y}\{\mathbf{i}_{123}\} = [K_2]\{\mathbf{i}_{123}\}, [K_2] \equiv \begin{bmatrix} 0 & k_4 & -k_{62} \\ -k_4 & 0 & -k_2 \\ k_{62} & k_2 & 0 \end{bmatrix} = \frac{\partial[T]}{\partial y}[T]^{\mathrm{T}} + [T][K_2^0][T]^{\mathrm{T}}, \quad (18)$$

where $[K_1]$ and $[K_2]$ are the deformed curvature matrices and the deformed curvatures are given by

$$k_1 \equiv -\frac{\partial \mathbf{i}_1}{\partial x}\cdot\mathbf{i}_3 = -T_{1mx}T_{3m} - T_{21}k_{61}^0 + T_{22}k_1^0 + T_{23}k_5^0$$

$$k_2 \equiv -\frac{\partial \mathbf{i}_2}{\partial y}\cdot\mathbf{i}_3 = -T_{2my}T_{3m} + T_{11}k_2^0 - T_{12}k_{62}^0 - T_{13}k_4^0$$

$$k_{61} \equiv -\frac{\partial \mathbf{i}_2}{\partial x}\cdot\mathbf{i}_3 = -T_{2mx}T_{3m} + T_{11}k_{61}^0 - T_{12}k_1^0 - T_{13}k_5^0$$

$$k_{62} \equiv -\frac{\partial \mathbf{i}_1}{\partial y}\cdot\mathbf{i}_3 = -T_{1my}T_{3m} - T_{21}k_2^0 + T_{22}k_{62}^0 + T_{23}k_4^0$$

$$k_5 \equiv \frac{\partial \mathbf{i}_1}{\partial x}\cdot\mathbf{i}_2 = T_{1mx}T_{2m} - T_{31}k_{61}^0 + T_{32}k_1^0 + T_{33}k_5^0$$

$$k_4 \equiv -\frac{\partial \mathbf{i}_2}{\partial y}\cdot \mathbf{i}_1 = -T_{2my}T_{1m} - T_{31}k_2^0 + T_{32}k_{62}^0 + T_{33}k_4^0. \tag{19}$$

Here, $\mathbf{i}_1 = \mathbf{i}_2 \times \mathbf{i}_3$, $\mathbf{i}_2 = \mathbf{i}_3 \times \mathbf{i}_1$, $\mathbf{i}_3 = \mathbf{i}_1 \times \mathbf{i}_2$, and eqn (15a) are used, and $T_{1mx}T_{3m} \equiv \Sigma_{m=1}^3 T_{3m}\,\partial T_{1m}/\partial x$. We note that the curvatures in eqn (19) do not represent real curvatures because the deformed dx (dy) is not along the $\mathbf{i}_1$ ($\mathbf{i}_2$) direction because $\gamma_{61} \neq 0$ ($\gamma_{62} \neq 0$). If $\gamma_{61} = \gamma_{62} = 0$, the curvatures are the normalized, not the real, curvatures because the differentiations in eqn (19) are taken with respect to the undeformed element lengths dx and dy, not the deformed lengths $(1+e_1)\,\mathrm{d}x$ and $(1+e_2)\,\mathrm{d}y$. When $\gamma_{61} = \gamma_{62} = e_1 = e_2 = 0$, k_1 and k_2 denote the bending curvatures with respect to the axes η and $-\xi$; k_{61} and k_{62} are the twisting curvatures with respect to the axes $-\xi$ and η, respectively; and k_4 and k_5 are the "drilling" or "spiral" curvatures, with respect to the axis ζ, of the axes η and ξ, respectively.

2.2. *Variations of in-plane strains and curvatures*

Using eqns (10) and (11) and the fact that $|\mathbf{i}_{\hat{1}}| = |\mathbf{i}_{\hat{2}}| = 1$, we obtain the variation of the in-plane strains e_1 and e_2 as

$$\delta e_1 = \hat{T}_{11}\delta t_{11} + \hat{T}_{12}\delta t_{12} + \hat{T}_{13}\delta t_{13} \tag{20}$$

$$\delta e_2 = \hat{T}_{21}\delta t_{21} + \hat{T}_{22}\delta t_{22} + \hat{T}_{23}\delta t_{23}, \tag{21}$$

where

$$\begin{aligned}
\delta t_{11} &= \delta(1+u_x - vk_5^0 + wk_1^0) = \delta u_x - k_5^0\delta v + k_1^0\delta w \\
\delta t_{12} &= \delta(v_x + uk_5^0 + wk_{61}^0) = \delta v_x + k_5^0\delta u + k_{61}^0\delta w \\
\delta t_{13} &= \delta(w_x - uk_1^0 - vk_{61}^0) = \delta w_x - k_1^0\delta u - k_{61}^0\delta v \\
\delta t_{21} &= \delta(u_y - vk_4^0 + wk_{62}^0) = \delta u_y - k_4^0\delta v + k_{62}^0\delta w \\
\delta t_{22} &= \delta(1+v_y + uk_4^0 + wk_2^0) = \delta v_y + k_4^0\delta u + k_2^0\delta w \\
\delta t_{23} &= \delta(w_y - uk_{62}^0 - vk_2^0) = \delta w_y - k_{62}^0\delta u - k_2^0\delta v.
\end{aligned} \tag{22}$$

It follows from eqns (8)–(11) that

$$\delta \mathbf{i}_{\hat{1}} = \frac{1}{1+e_1}(\mathbf{j}_1\delta t_{11} + \mathbf{j}_2\delta t_{12} + \mathbf{j}_3\delta t_{13} - \mathbf{i}_{\hat{1}}\delta e_1) \tag{23a}$$

$$\delta \mathbf{i}_{\hat{2}} = \frac{1}{1+e_2}(\mathbf{j}_1\delta t_{21} + \mathbf{j}_2\delta t_{22} + \mathbf{j}_3\delta t_{23} - \mathbf{i}_{\hat{2}}\delta e_2). \tag{23b}$$

Taking the variation of eqn (12a) and using eqns (23), (8), (9), (20), and (21), we obtain

$$\delta\gamma_6 = \frac{(\hat{T}_{21} - \sin\gamma_6\hat{T}_{11})\delta t_{11} + (\hat{T}_{22} - \sin\gamma_6\hat{T}_{12})\delta t_{12} + (\hat{T}_{23} - \sin\gamma_6\hat{T}_{13})\delta t_{13}}{\cos\gamma_6(1+e_1)} + \frac{(\hat{T}_{11} - \sin\gamma_6\hat{T}_{21})\delta t_{21} + (\hat{T}_{12} - \sin\gamma_6\hat{T}_{22})\delta t_{22} + (\hat{T}_{13} - \sin\gamma_6\hat{T}_{23})\delta t_{23}}{\cos\gamma_6(1+e_2)}. \tag{24}$$

Taking the variation of eqn (12b) and using the fact that $\delta\gamma_6 = \delta\gamma_{61} + \delta\gamma_{62}$, we obtain

$$\delta\gamma_{61} = \frac{(1+e_2)\cos\gamma_{62}\delta\gamma_6 - \sin\gamma_{61}\delta e_1 + \sin\gamma_{62}\delta e_2}{(1+e_1)\cos\gamma_{61} + (1+e_2)\cos\gamma_{62}} \tag{25a}$$

$$\delta\gamma_{62} = \frac{(1+e_1)\cos\gamma_{61}\delta\gamma_6 + \sin\gamma_{61}\delta e_1 - \sin\gamma_{62}\delta e_2}{(1+e_1)\cos\gamma_{61} + (1+e_2)\cos\gamma_{62}}. \tag{25b}$$

In the variational formulation of a structural system, the variations of strains and therefore curvatures are always involved. To obtain the variations of curvatures (i.e. δk_j), the concept of orthogonal virtual rotations is needed. Because the variations of the unit vectors $\mathbf{i}_k$ are due to the virtual rotations of the observed shell element,

$$\delta\{\mathbf{i}_{123}\} = [\delta\theta]\{\mathbf{i}_{123}\}, \quad [\delta\theta] \equiv \begin{bmatrix} 0 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 0 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 0 \end{bmatrix}, \tag{26}$$

where $\delta\theta_1$, $\delta\theta_2$, and $\delta\theta_3$ are the virtual rigid-body rotations of the observed shell element with respect to the axes ξ, η, and ζ, respectively. We note that $\delta\theta_i$ are infinitesimal rotations and hence they are vector quantities. Moreover, $\delta\theta_i$ are along three perpendicular directions and hence they are mutually independent. Taking the variations of curvatures defined in eqns (19) and using eqns (17), (18), and (26), Pai and Nayfeh (1991, 1994a) showed that

$$\begin{Bmatrix} -\delta k_{61} \\ \delta k_1 \\ \delta k_5 \end{Bmatrix} = \frac{\partial}{\partial x}\begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{Bmatrix} - [K_1]\begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{Bmatrix} \tag{27a}$$

$$\begin{Bmatrix} -\delta k_2 \\ \delta k_{62} \\ \delta k_4 \end{Bmatrix} = \frac{\partial}{\partial y}\begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{Bmatrix} - [K_2]\begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{Bmatrix}. \tag{27b}$$

Using eqns (26), (15b), and (23) and the fact that $\mathbf{i}_3 \cdot \mathbf{i}_{\hat{1}} = \mathbf{i}_3 \cdot \mathbf{i}_{\hat{2}} = 0$, we obtain

$$\begin{aligned} \delta\theta_1 &= \delta\mathbf{i}_2 \cdot \mathbf{i}_3 = \frac{\cos\gamma_{61}}{\cos\gamma_6}\delta\mathbf{i}_{\hat{2}} \cdot \mathbf{i}_3 - \frac{\sin\gamma_{62}}{\cos\gamma_6}\delta\mathbf{i}_{\hat{1}} \cdot \mathbf{i}_3 \\ &= \frac{\cos\gamma_{61}}{\cos\gamma_6(1+e_2)}(T_{31}\delta t_{21} + T_{32}\delta t_{22} + T_{33}\delta t_{23}) \\ &\quad - \frac{\sin\gamma_{62}}{\cos\gamma_6(1+e_1)}(T_{31}\delta t_{11} + T_{32}\delta t_{12} + T_{33}\delta t_{13}) \end{aligned} \tag{28}$$

$$\begin{aligned} \delta\theta_2 &= -\delta\mathbf{i}_1 \cdot \mathbf{i}_3 = \frac{\sin\gamma_{61}}{\cos\gamma_6}\delta\mathbf{i}_{\hat{2}} \cdot \mathbf{i}_3 - \frac{\cos\gamma_{62}}{\cos\gamma_6}\delta\mathbf{i}_{\hat{1}} \cdot \mathbf{i}_3 \\ &= \frac{\sin\gamma_{61}}{\cos\gamma_6(1+e_2)}(T_{31}\delta t_{21} + T_{32}\delta t_{22} + T_{33}\delta t_{23}) \\ &\quad - \frac{\cos\gamma_{62}}{\cos\gamma_6(1+e_1)}(T_{31}\delta t_{11} + T_{32}\delta t_{12} + T_{33}\delta t_{13}). \end{aligned} \tag{29}$$

Using eqns (26), (15b), and (23), we obtain

$$
\begin{aligned}
\delta\theta_3 &= \tfrac{1}{2}(\delta\mathbf{i}_1\cdot\mathbf{i}_2-\delta\mathbf{i}_2\cdot\mathbf{i}_1)=\tfrac{1}{2}(\delta\gamma_{62}-\delta\gamma_{61})+\frac{1}{2\cos\gamma_6}(\delta\mathbf{i}_1\cdot\mathbf{i}_2-\delta\mathbf{i}_2\cdot\mathbf{i}_1)\\
&=\frac{(\hat{T}_{21}-\sin\gamma_6\hat{T}_{11})\delta t_{11}+(\hat{T}_{22}-\sin\gamma_6\hat{T}_{12})\delta t_{12}+(\hat{T}_{23}-\sin\gamma_6\hat{T}_{13})\delta t_{13}}{2\cos\gamma_6(1+e_1)}\\
&\quad-\frac{(\hat{T}_{11}-\sin\gamma_6\hat{T}_{21})\delta t_{21}+(\hat{T}_{12}-\sin\gamma_6\hat{T}_{22})\delta t_{22}+(\hat{T}_{13}-\sin\gamma_6\hat{T}_{23})\delta t_{23}}{2\cos\gamma_6(1+e_2)}\\
&\quad+\tfrac{1}{2}(\delta\gamma_{62}-\delta\gamma_{61}).
\end{aligned}\tag{30}
$$

Hence, δe_1, δe_2, $\delta\gamma_6$, $\delta\gamma_{61}$, $\delta\gamma_{62}$, $\delta\theta_1$, $\delta\theta_2$, and $\delta\theta_3$ can be represented in terms of δu, δv, δw, $\delta u_{,x}$, $\delta v_{,x}$, $\delta w_{,x}$, $\delta u_{,y}$, $\delta v_{,y}$, and $\delta w_{,y}$. Moreover, the variations of curvatures δk_j can be represented in terms of δu, δv, δw, $\delta u_{,x}$, $\delta v_{,x}$, $\delta w_{,x}$, $\delta u_{,y}$, $\delta v_{,y}$, $\delta w_{,y}$, $\delta u_{,xx}$, $\delta v_{,xx}$, $\delta w_{,xx}$, $\delta u_{,yy}$, $\delta v_{,yy}$, $\delta w_{,yy}$, $\delta u_{,xy}$, $\delta v_{,xy}$, and $\delta w_{,xy}$.

3. SHEAR WARPINGS AND JAUMANN STRAINS

3.1. *Shear warping functions*

To include shear deformations in the modeling of a general anisotropic laminated shell consisting of N layers, each layer needs an assumed displacement field because the material property is not uniform through the thickness. To account for the transverse normal strain as well as transverse shear deformations, we modify the layer-wise local displacement field used by Pai and Nayfeh (1994a) and assume that the local displacement vector $\mathbf{u}$ (with respect to the coordinate system $\xi\eta\zeta$) of the ith lamina has the form

$$\mathbf{u}=u_1^{(i)}\mathbf{i}_1+u_2^{(i)}\mathbf{i}_2+u_3^{(i)}\mathbf{i}_3,\tag{31}$$

where

$$
\begin{aligned}
u_1^{(i)} &= u_1^0(x,y,t)+z[\theta_2(x,y,t)-\theta_2^0(x,y)]+\gamma_5 z+\alpha_1^{(i)}(x,y,t)z^2+\beta_1^{(i)}(x,y,t)z^3\\
u_2^{(i)} &= u_2^0(x,y,t)-z[\theta_1(x,y,t)-\theta_1^0(x,y)]+\gamma_4 z+\alpha_2^{(i)}(x,y,t)z^2+\beta_2^{(i)}(x,y,t)z^3\\
u_3^{(i)} &= u_3^0(x,y,t)+\alpha_3^{(i)}(x,y,t)z+\beta_3^{(i)}(x,y,t)z^2.
\end{aligned}\tag{32}
$$

Here, t denotes time, $u_j^0(j=1,2,3)$ are the displacements (with respect to the local coordinate system $\xi\eta\zeta$) of a point which is located at $(x,y,0)$ before deformation, $\gamma_4(x,y,t)$ and $\gamma_5(x,y,t)$ are the transverse shear rotation angles at the reference surface, θ_1 and θ_2 are the rotation angles of the normal of the reference surface with respect to the axes ξ and η, respectively, and θ_1^0 and θ_2^0 are the corresponding initial rotation angles. Moreover, $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are functions to be determined by using the continuity conditions of displacements and interlaminar stresses and the stress conditions on the bonding surfaces. Because the frame $\xi\eta\zeta$ is a local coordinate system and the plane $\xi-\eta$ is tangential to the deformed reference surface, we have

$$u_1^0=u_2^0=u_3^0=\theta_1^0=\theta_2^0=\theta_1=\theta_2=\partial u_3^0/\partial x=\partial u_3^0/\partial y=0.\tag{33}$$

To use the local displacement field (31) to derive in-plane strains and deformed curvatures, one needs to consider a point which is very close to but not the same as the observed point and its values of those terms in eqn (33) are small but not zero. Then one takes the derivatives of the local displacement vector and uses eqn (33) to derive in-plane strains and curvatures. This is called the concept of local displacements [see Appendix B of Pai and Nayfeh (1994a)]. It follows from Fig. 1(b) and the concept of local displacements that (Pai and Palazotto, 1994a)

$$\frac{\partial u_2^0}{(1+e_1)\,\partial x} = \sin\gamma_{61}, \quad \frac{\partial u_1^0}{(1+e_2)\,\partial y} = \sin\gamma_{62} \tag{34a}$$

$$\frac{\partial u_1^0}{\partial x} = (1+e_1)\cos\gamma_{61} - 1, \quad \frac{\partial u_2^0}{\partial y} = (1+e_2)\cos\gamma_{62} - 1 \tag{34b}$$

$$\frac{\partial \theta_2^0}{\partial x} = -\frac{\partial \mathbf{j}_1}{\partial x}\cdot\mathbf{j}_3 = k_1^0, \quad \frac{\partial \theta_1^0}{\partial y} = -k_2^0, \quad \frac{\partial \theta_1^0}{\partial x} = -k_{61}^0, \quad \frac{\partial \theta_2^0}{\partial y} = k_{62}^0 \tag{34c}$$

$$\frac{\partial \theta_2}{\partial x} = -\frac{\partial \mathbf{i}_1}{\partial x}\cdot\mathbf{i}_3 = k_1, \quad \frac{\partial \theta_1}{\partial y} = -k_2, \quad \frac{\partial \theta_1}{\partial x} = -k_{61}, \quad \frac{\partial \theta_2}{\partial y} = k_{62}. \tag{34d}$$

Taking the derivatives of eqns (31) and (32) and using eqns (33) and (34), we obtain

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial x} &= \frac{\partial u_1}{\partial x}\mathbf{i}_1 + \frac{\partial u_2}{\partial x}\mathbf{i}_2 + \frac{\partial u_3}{\partial x}\mathbf{i}_3 + u_1\frac{\partial \mathbf{i}_1}{\partial x} + u_2\frac{\partial \mathbf{i}_2}{\partial x} + u_3\frac{\partial \mathbf{i}_3}{\partial x}\\ &= [(1+e_1)\cos\gamma_{61} - 1 + z(k_1 - k_1^0) + G_{1x} - k_5 G_2 + k_1 G_3]\mathbf{i}_1\\ &\quad + [(1+e_1)\sin\gamma_{61} + z(k_{61} - k_{61}^0) + G_{2x} + k_5 G_1 + k_{61} G_3]\mathbf{i}_2\\ &\quad + [G_{3x} - k_1 G_1 - k_{61} G_2]\mathbf{i}_3\end{aligned} \tag{35a}$$

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial y} &= \frac{\partial u_1}{\partial y}\mathbf{i}_1 + \frac{\partial u_2}{\partial y}\mathbf{i}_2 + \frac{\partial u_3}{\partial y}\mathbf{i}_3 + u_1\frac{\partial \mathbf{i}_1}{\partial y} + u_2\frac{\partial \mathbf{i}_2}{\partial y} + u_3\frac{\partial \mathbf{i}_3}{\partial y}\\ &= [(1+e_2)\sin\gamma_{62} + z(k_{62} - k_{62}^0) + G_{1y} - k_4 G_2 + k_{62} G_3]\mathbf{i}_1\\ &\quad + [(1+e_2)\cos\gamma_{62} - 1 + z(k_2 - k_2^0) + G_{2y} + k_4 G_1 + k_2 G_3]\mathbf{i}_2\\ &\quad + [G_{3y} - k_{62} G_1 - k_2 G_2]\mathbf{i}_3\end{aligned} \tag{35b}$$

$$\frac{\partial \mathbf{u}}{\partial z} = \frac{\partial u_1}{\partial z}\mathbf{i}_1 + \frac{\partial u_2}{\partial z}\mathbf{i}_2 + \frac{\partial u_3}{\partial z}\mathbf{i}_3 = G_{1z}\mathbf{i}_1 + G_{2z}\mathbf{i}_2 + G_{3z}\mathbf{i}_3, \tag{35c}$$

where

$$G_1 \equiv \gamma_5 z + \alpha_1^{(i)} z^2 + \beta_1^{(i)} z^3,\ G_2 \equiv \gamma_4 z + \alpha_2^{(i)} z^2 + \beta_2^{(i)} z^3,\ G_3 \equiv \alpha_3^{(i)} z + \beta_3^{(i)} z^2. \tag{36}$$

Hence, Jaumann strains B_{mn} are obtained as

$$\begin{aligned}B_{11}^{(i)} &= \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{i}_1 = (1+e_1)\cos\gamma_{61} - 1 + z(k_1 - k_1^0) + G_{1x} - k_5 G_2 + k_1 G_3\\ B_{22}^{(i)} &= \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{i}_2 = (1+e_2)\cos\gamma_{62} - 1 + z(k_2 - k_2^0) + G_{2y} + k_4 G_1 + k_2 G_3\\ B_{33}^{(i)} &= \frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{i}_3 = G_{3z} = \alpha_3^{(i)} + 2z\beta_3^{(i)}\\ 2B_{23}^{(i)} &= \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{i}_3 + \frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{i}_2 = G_{2z} - k_{62} G_1 - k_2 G_2 + G_{3y}\\ 2B_{13}^{(i)} &= \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{i}_3 + \frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{i}_1 = G_{1z} - k_1 G_1 - k_{61} G_2 + G_{3x}\\ 2B_{12}^{(i)} &= \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{i}_2 + \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{i}_1\\ &= (1+e_1)\sin\gamma_{61} + (1+e_2)\sin\gamma_{62} + z(k_6 - k_6^0)\\ &\quad + G_{1y} + G_{2x} + k_5 G_1 - k_4 G_2 + k_6 G_3,\end{aligned} \tag{37}$$

where $k_6 \equiv k_{61}+k_{62}$ and $k_6^0 \equiv k_{61}^0+k_{62}^0$. We point out here that we have neglected the "trapezoidal-cross-section" effects in the strains. Moreover, because G_3 is usually small, especially for thin shells, one can neglect G_3 in all strain-displacement expressions except in that of $B_{33}^{(i)}$.

To derive approximate analytical shear warping functions, we replace the deformed curvatures with the initial curvatures in eqn (37) and neglect the influence of G_{3x} and G_{3y} to rewrite the transverse shear strains as

$$
\begin{aligned}
2B_{23}^{(i)} &= \gamma_4(1-k_2^0 z)-\gamma_5(k_{62}^0 z) \\
&\quad -\alpha_1^{(i)}(k_{62}^0 z^2)-\beta_1^{(i)}(k_{62}^0 z^3)+\alpha_2^{(i)}(2z-k_2^0 z^2)+\beta_2^{(i)}(3z^2-k_2^0 z^3) \\
2B_{13}^{(i)} &= -\gamma_4(k_{61}^0 z)+\gamma_5(1-k_1^0 z) \\
&\quad +\alpha_1^{(i)}(2z-k_1^0 z^2)+\beta_1^{(i)}(3z^2-k_1^0 z^3)-\alpha_2^{(i)}(k_{61}^0 z^2)-\beta_2^{(i)}(k_{61}^0 z^3).
\end{aligned} \tag{38}
$$

Using tensor transformations, we obtain the transformed stiffness matrix $[\bar{Q}^{(i)}]$ for the ith lamina from its principal stiffness matrix $[\bar{Q}^{(i)}]$ and its ply angle (measured with respect to the axis x), and have the stress–strain relations for the ith lamina as

$$\{J^{(i)}\} = [\bar{Q}^{(i)}]\{B^{(i)}\},\ \{J_1^{(i)}\} = [\bar{Q}_1^{(i)}]\{B_1^{(i)}\},\ \{J_2^{(i)}\} = [\bar{Q}_2^{(i)}]\{B_2^{(i)}\}, \tag{39a}$$

where

$$
[\bar{Q}^{(i)}] \equiv \begin{bmatrix} [\bar{Q}_1^{(i)}] & [0] \\ [0] & [\bar{Q}_2^{(i)}] \end{bmatrix}
$$

$$
[\bar{Q}_1^{(i)}] \equiv \begin{bmatrix} \bar{Q}_{11}^{(i)} & \bar{Q}_{12}^{(i)} & \bar{Q}_{13}^{(i)} & \bar{Q}_{16}^{(i)} \\ \bar{Q}_{12}^{(i)} & \bar{Q}_{22}^{(i)} & \bar{Q}_{23}^{(i)} & \bar{Q}_{26}^{(i)} \\ \bar{Q}_{13}^{(i)} & \bar{Q}_{23}^{(i)} & \bar{Q}_{33}^{(i)} & \bar{Q}_{36}^{(i)} \\ \bar{Q}_{16}^{(i)} & \bar{Q}_{26}^{(i)} & \bar{Q}_{36}^{(i)} & \bar{Q}_{66}^{(i)} \end{bmatrix},\quad [\bar{Q}_2^{(i)}] \equiv \begin{bmatrix} \bar{Q}_{44}^{(i)} & \bar{Q}_{45}^{(i)} \\ \bar{Q}_{45}^{(i)} & \bar{Q}_{55}^{(i)} \end{bmatrix} \tag{39b}
$$

$$
\begin{aligned}
\{J^{(i)}\} &= \{\{J_1^{(i)}\}^{\mathrm{T}},\{J_2^{(i)}\}^{\mathrm{T}}\}^{\mathrm{T}},\ \{B^{(i)}\} = \{\{B_1^{(i)}\}^{\mathrm{T}},\{B_2^{(i)}\}^{\mathrm{T}}\}^{\mathrm{T}} \\
\{J_1^{(i)}\} &= \{J_{11}^{(i)},J_{22}^{(i)},J_{33}^{(i)},J_{12}^{(i)}\}^{\mathrm{T}},\ \{B_1^{(i)}\} = \{B_{11}^{(i)},B_{22}^{(i)},B_{33}^{(i)},2B_{12}^{(i)}\}^{\mathrm{T}} \\
\{J_2^{(i)}\} &= \{J_{23}^{(i)},J_{13}^{(i)}\}^{\mathrm{T}},\ \{B_2^{(i)}\} = \{2B_{23}^{(i)},2B_{13}^{(i)}\}^{\mathrm{T}}.
\end{aligned} \tag{39c}
$$

It is assumed that there is no delamination, and hence the in-plane displacements u_1 and u_2 and interlaminar shear stresses J_{13} and J_{23} are continuous across the interface of two adjacent laminae. Moreover, it is assumed that there are no applied shear loads on the bounding surfaces and hence $J_{13} = J_{23} = B_{13} = B_{23} = 0$ at the $z = z_1$ and $z = z_{N+1}$ planes, where N is the total number of layers. Hence, we have

$$
\begin{aligned}
B_{23}^{(1)}(x,y,z_1,t) &= 0 \\
B_{13}^{(1)}(x,y,z_1,t) &= 0 \\
u_1^{(i)}(x,y,z_{i+1},t)-u_1^{(i+1)}(x,y,z_{i+1},t) &= 0 \quad \text{for } i = 1,\ldots,N-1 \\
u_2^{(i)}(x,y,z_{i+1},t)-u_2^{(i-1)}(x,y,z_{i-1},t) &= 0 \quad \text{for } i = 1,\ldots,N-1 \\
J_{23}^{(i)}(x,y,z_{i+1},t)-J_{23}^{(i+1)}(x,y,z_{i+1},t) &= 0 \quad \text{for } i = 1,\ldots,N-1 \\
J_{13}^{(i)}(x,y,z_{i+1},t)-J_{13}^{(i+1)}(x,y,z_{i+1},t) &= 0 \quad \text{for } i = 1,\ldots,N-1 \\
B_{23}^{(N)}(x,y,z_{N+1},t) &= 0 \\
B_{13}^{(N)}(x,y,z_{N+1},t) &= 0.
\end{aligned} \tag{40}
$$

These $4N$ algebraic equations can be used to determine $4N$ unknowns (i.e. $\alpha_1^{(i)}, \alpha_2^{(i)}, \beta_1^{(i)}, \beta_2^{(i)}$ for $i = 1, \ldots, N$) in terms of γ_4 and γ_5 as

$$\alpha_1^{(i)} = a_{14}^{(i)}\gamma_4 + a_{15}^{(i)}\gamma_5, \quad \alpha_2^{(i)} = a_{24}^{(i)}\gamma_4 + a_{25}^{(i)}\gamma_5,$$
$$\beta_1^{(i)} = b_{14}^{(i)}\gamma_4 + b_{15}^{(i)}\gamma_5, \quad \beta_2^{(i)} = b_{24}^{(i)}\gamma_4 + b_{25}^{(i)}\gamma_5, \tag{41}$$

where $a_{kl}^{(i)}$ and $b_{kl}^{(i)}$ are functions of z_j, $\bar{Q}_{44}^{(i)}$, $\bar{Q}_{45}^{(i)}$, and $\bar{Q}_{55}^{(i)}$. Hence,

$$G_1 = \gamma_5 g_{15}^{(i)} + \gamma_4 g_{14}^{(i)}, \quad G_2 = \gamma_4 g_{24}^{(i)} + \gamma_5 g_{25}^{(i)}, \tag{42a,b}$$

where $g_{14}^{(i)}, g_{15}^{(i)}, g_{24}^{(i)}$, and $g_{25}^{(i)}$ are polynominal functions of z, defined as

$$g_{14}^{(i)} \equiv a_{14}^{(i)}z^2 + b_{14}^{(i)}z^3, \quad g_{15}^{(i)} \equiv z + a_{15}^{(i)}z^2 + b_{15}^{(i)}z^3,$$
$$g_{24}^{(i)} \equiv z + a_{24}^{(i)}z^2 + b_{24}^{(i)}z^3, \quad g_{25}^{(i)} \equiv a_{25}^{(i)}z^2 + b_{25}^{(i)}z^3. \tag{43}$$

Substituting eqns (42), (43), (37), (39), and (32) into eqns (40) and then setting the coefficients of γ_4 and γ_5 to zero yield $8N$ algebraic equations, which can be used to determine the $8N$ unknowns—$a_{14}^{(i)}, a_{15}^{(i)}, a_{24}^{(i)}, a_{25}^{(i)}, b_{14}^{(i)}, b_{15}^{(i)}, b_{24}^{(i)}, b_{25}^{(i)}$, for $i = 1, \ldots, N$ (Pai and Nayfeh, 1994a).

Substituting eqns (42) and (43) into eqn (37) yields

$$\{B_{11}^{(i)}, B_{22}^{(i)}, 2B_{12}^{(i)}\}^{\mathrm{T}} = [S_0^{(i)}]\{\psi\}, \tag{44}$$

where

$$[S_0^{(i)}] \equiv \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 & g_{14}^{(i)} & 0 & g_{15}^{(i)} & 0 & -k_5 g_{24}^{(i)} & -k_5 g_{25}^{(i)} \\ 0 & 1 & 0 & 0 & z & 0 & 0 & g_{24}^{(i)} & 0 & g_{25}^{(i)} & k_4 g_{14}^{(i)} & k_4 g_{15}^{(i)} \\ 0 & 0 & 1 & 0 & 0 & z & g_{24}^{(i)} & g_{14}^{(i)} & g_{25}^{(i)} & g_{15}^{(i)} & k_5 g_{14}^{(i)} - k_4 g_{24}^{(i)} & k_5 g_{15}^{(i)} - k_4 g_{25}^{(i)} \end{bmatrix} \tag{45}$$

$$\{\psi\} \equiv \{(1+e_1)\cos\gamma_{61} - 1, \quad (1+e_2)\cos\gamma_{62} - 1, \quad (1+e_1)\sin\gamma_{61} + (1+e_2)\sin\gamma_{62},$$
$$k_1 - k_1^0, k_2 - k_2^0, k_6 - k_6^0, \gamma_{4x}, \gamma_{4y}, \gamma_{5x}, \gamma_{5y}, \gamma_4, \gamma_5\}^{\mathrm{T}}. \tag{46}$$

Then, it follows from eqns (39a, b), (44), (37), (32), and (33) that $\alpha_3^{(i)}$ and $\beta_3^{(i)}$ can be determined by using the normal stress conditions on the deformed bonding surfaces and the continuity conditions of $u_3^{(i)}$ and $J_{33}^{(i)}$ as

$$\bar{Q}_{33}^{(1)}(\alpha_3^{(1)} + 2\beta_3^{(1)}z_1) = -\{\bar{Q}_{13}^{(1)}, \bar{Q}_{23}^{(1)}, \bar{Q}_{36}^{(1)}\}[S_0^{(1)}]_{z_1}\{\psi\} \tag{47}$$

$$\bar{Q}_{33}^{(i)}(\alpha_3^{(i)} + 2\beta_3^{(i)}z_{i+1}) - \bar{Q}_{33}^{(i+1)}(\alpha_3^{(i+1)} + 2\beta_3^{(i+1)}z_{i+1})$$
$$= (\{\bar{Q}_{13}^{(i+1)}, \bar{Q}_{23}^{(i+1)}, \bar{Q}_{36}^{(i+1)}\}[S_0^{(i+1)}]_{z_{i+1}} - \{\bar{Q}_{13}^{(i)}, \bar{Q}_{23}^{(i)}, \bar{Q}_{36}^{(i)}\}[S_0^{(i)}]_{z_{i+1}})\{\psi\}$$
$$\alpha_3^{(i)}z_{i+1} + \beta_3^{(i)}z_{i+1}^2 - \alpha_3^{(i+1)}z_{i+1} - \beta_3^{(i+1)}z_{i+1}^2 = 0 \quad \text{for} \quad i = 1, \ldots, N-1 \tag{48}$$

$$\bar{Q}_{33}^{(N)}(\alpha_3^{(N)} + 2\beta_3^{(N)}z_{N+1}) = -\{\bar{Q}_{13}^{(N)}, \bar{Q}_{23}^{(N)}, \bar{Q}_{36}^{(N)}\}[S_0^{(N)}]_{z_{N+1}}\{\psi\}, \tag{49}$$

where we assume that there are no externally applied normal stresses on the deformed bonding surfaces. Equations (47)–(49) are $2N$ equations, which can be used to solve for the $2N$ unknowns $\alpha_3^{(i)}$ and $\beta_3^{(i)}$, $i = 1, \ldots, N$, in terms of $\{\psi\}$ as

$$\begin{Bmatrix} \alpha_3^{(i)} \\ \beta_3^{(i)} \end{Bmatrix} = \begin{bmatrix} a_{30}^{(i)} & a_{31}^{(i)} & a_{32}^{(i)} \\ b_{30}^{(i)} & b_{31}^{(i)} & b_{32}^{(i)} \end{bmatrix} \begin{Bmatrix} (1+e_1)\cos\gamma_{61} - 1 \\ (1+e_2)\cos\gamma_{62} - 1 \\ (1+e_1)\sin\gamma_{61} + (1+e_2)\sin\gamma_{62} \end{Bmatrix}$$

$$+ \begin{bmatrix} a_{33}^{(i)} & a_{34}^{(i)} & a_{35}^{(i)} \\ b_{33}^{(i)} & b_{34}^{(i)} & b_{35}^{(i)} \end{bmatrix} \begin{Bmatrix} k_1 - k_1^0 \\ k_2 - k_2^0 \\ k_6 - k_6^0 \end{Bmatrix}$$

$$+ \begin{bmatrix} a_{36}^{(i)} & a_{37}^{(i)} & a_{38}^{(i)} & a_{39}^{(i)} \\ b_{36}^{(i)} & b_{37}^{(i)} & b_{38}^{(i)} & b_{39}^{(i)} \end{bmatrix} \begin{Bmatrix} \gamma_{4x} \\ \gamma_{4y} \\ \gamma_{5x} \\ \gamma_{5y} \end{Bmatrix} + \begin{bmatrix} a_{41}^{(i)} & a_{42}^{(i)} & a_{43}^{(i)} & a_{44}^{(i)} \\ b_{41}^{(i)} & b_{42}^{(i)} & b_{43}^{(i)} & b_{44}^{(i)} \end{bmatrix} \begin{Bmatrix} \gamma_4 k_4 \\ \gamma_4 k_5 \\ \gamma_5 k_4 \\ \gamma_5 k_5 \end{Bmatrix}. \tag{50}$$

3.2. *Strain-displacement relations*

Substituting eqns (42) and (43) into eqn (37) and neglecting G_3, we obtain the Jaumann strains $B_{ij}^{(i)}$ for the ith lamina as

$$\begin{aligned}
B_{11}^{(i)} &= (1+e_1)\cos\gamma_{61} - 1 + z(k_1 - k_1^0) + \gamma_{5x} g_{15}^{(i)} + \gamma_{4x} g_{14}^{(i)} - k_5(\gamma_4 g_{24}^{(i)} + \gamma_5 g_{25}^{(i)}) \\
B_{22}^{(i)} &= (1+e_2)\cos\gamma_{62} - 1 + z(k_2 - k_2^0) + \gamma_{4y} g_{24}^{(i)} + \gamma_{5y} g_{25}^{(i)} + k_4(\gamma_5 g_{15}^{(i)} + \gamma_4 g_{14}^{(i)}) \\
B_{33}^{(i)} &= \alpha_3^{(i)} + 2z\beta_3^{(i)} \\
2B_{23}^{(i)} &= \gamma_4 g_{24z}^{(i)} + \gamma_5 g_{25z}^{(i)} - k_{62}(\gamma_5 g_{15}^{(i)} + \gamma_4 g_{14}^{(i)}) - k_2(\gamma_4 g_{24}^{(i)} + \gamma_5 g_{25}^{(i)}) \\
2B_{13}^{(i)} &= \gamma_5 g_{15z}^{(i)} + \gamma_4 g_{14z}^{(i)} - k_1(\gamma_5 g_{15}^{(i)} + \gamma_4 g_{14}^{(i)}) - k_{61}(\gamma_4 g_{24}^{(i)} + \gamma_5 g_{25}^{(i)}) \\
2B_{12}^{(i)} &= (1+e_1)\sin\gamma_{61} + (1+e_2)\sin\gamma_{62} + z(k_6 - k_6^0) \\
&\quad + \gamma_{5y} g_{15}^{(i)} + \gamma_{4y} g_{14}^{(i)} + \gamma_{4x} g_{24}^{(i)} + \gamma_{5x} g_{25}^{(i)} \\
&\quad + k_5(\gamma_5 g_{15}^{(i)} + \gamma_4 g_{14}^{(i)}) - k_4(\gamma_4 g_{24}^{(i)} + \gamma_5 g_{25}^{(i)}),
\end{aligned} \tag{51}$$

where $\alpha_3^{(i)}$ and $\beta_3^{(i)}$ are given by eqn (50). It follows from eqns (51) and (39c) that the Jaumann strains can be written in matrix forms as

$$\{B_1^{(i)}\} = [S_1^{(i)}]\{\psi_1\} + [S_3^{(i)}]\{\psi_3\} \tag{52a}$$

$$\{B_2^{(i)}\} = ([S_2^{(i)}] + [S_4^{(i)}])\{\psi_2\}, \tag{52b}$$

where

$$[S_1^{(i)}] \equiv \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 \\ g_{30}^{(i)} & g_{31}^{(i)} & g_{32}^{(i)} & g_{33}^{(i)} & g_{34}^{(i)} & g_{35}^{(i)} \\ 0 & 0 & 1 & 0 & 0 & z \end{bmatrix}$$

$$[S_3^{(i)}] \equiv \begin{bmatrix} g_{14}^{(i)} & 0 & g_{15}^{(i)} & 0 & -k_5 g_{24}^{(i)} & -k_5 g_{25}^{(i)} \\ 0 & g_{24}^{(i)} & 0 & g_{25}^{(i)} & k_4 g_{14}^{(i)} & k_4 g_{15}^{(i)} \\ g_{36}^{(i)} & g_{37}^{(i)} & g_{38}^{(i)} & g_{39}^{(i)} & k_4 g_{41}^{(i)} + k_5 g_{42}^{(i)} & k_4 g_{43}^{(i)} + k_5 g_{44}^{(i)} \\ g_{24}^{(i)} & g_{14}^{(i)} & g_{25}^{(i)} & g_{15}^{(i)} & k_5 g_{14}^{(i)} - k_4 g_{24}^{(i)} & k_5 g_{15}^{(i)} - k_4 g_{25}^{(i)} \end{bmatrix}$$

$$[S_2^{(i)}] \equiv \begin{bmatrix} g_{24z}^{(i)} & g_{25z}^{(i)} \\ g_{14z}^{(i)} & g_{15z}^{(i)} \end{bmatrix}, \quad [S_4^{(i)}] \equiv -\begin{bmatrix} k_2 & k_{62} \\ k_{61} & k_1 \end{bmatrix} \begin{bmatrix} g_{24}^{(i)} & g_{25}^{(i)} \\ g_{14}^{(i)} & g_{15}^{(i)} \end{bmatrix} \tag{53a}$$

$$\{\psi_1\} \equiv \{(1+e_1)\cos\gamma_{61}-1, (1+e_2)\cos\gamma_{62}-1,$$
$$(1+e_1)\sin\gamma_{61}+(1+e_2)\sin\gamma_{62}, k_1-k_1^0, k_2-k_2^0, k_6-k_6^0\}$$
$$\{\psi_2\} \equiv \{\gamma_4,\gamma_5\}^{\mathrm{T}}$$
$$\{\psi_3\} \equiv \{\gamma_{4x},\gamma_{4y},\gamma_{5x},\gamma_{5y},\gamma_4,\gamma_5\}^{\mathrm{T}} \tag{53b}$$

$$g_{3j}^{(i)} \equiv a_{3j}^{(i)}+2b_{3j}^{(i)}z, j=0,1,\ldots,9$$
$$g_{4j}^{(i)} \equiv a_{4j}^{(i)}+2b_{4j}^{(i)}z, j=1,\ldots,4. \tag{53c}$$

The strain-displacement relations shown in eqns (52) can be combined as

$$\{B^{(i)}\} = [S^{(i)}]\{\psi\}, \tag{54a}$$

where $\{\psi\}$ is shown in eqn (46) and

$$[S^{(i)}] \equiv \begin{bmatrix} [S_1^{(i)}] & [S_3^{(i)}] \\ [0] & [S_2^{(i)}]+[S_4^{(i)}] \end{bmatrix}, \tag{54b}$$

where [0] is a 2×10 null matrix.

4. FINITE-ELEMENT FORMULATION

The dynamic version of the principle of virtual work states (Washizu, 1982)

$$\int_0^t (\delta K_{\mathrm{E}}-\delta\Pi+\delta W_{\mathrm{nc}})\,\mathrm{d}t = 0, \tag{55}$$

where Π denotes the elastic energy, K_{E} denotes the kinetic energy, and W_{nc} denotes the nonconservative energy, which includes the energy due to external distributed and/or concentrated loads and dampings.

4.1. *Elastic energy*

Because the elastic energy Π is due to relative displacements among material particles, its variation $\delta\Pi$ is given by

$$\delta\Pi = \int_{t^0}\left(\mathbf{f}_1\cdot\mathbf{i}_n\delta\frac{\partial\mathbf{u}}{\partial x_1}\mathrm{d}x_1\cdot\mathbf{i}_n+\mathbf{f}_2\cdot\mathbf{i}_n\delta\frac{\partial\mathbf{u}}{\partial x_2}\mathrm{d}x_2\cdot\mathbf{i}_n+\mathbf{f}_3\cdot\mathbf{i}_n\delta\frac{\partial\mathbf{u}}{\partial x_3}\mathrm{d}x_3\cdot\mathbf{i}_n\right),$$

where $\mathbf{f}_1$ is the force vector acting on the deformed surface of the undeformed area $\mathrm{d}x_2\mathrm{d}x_3$, as shown in Fig. 2. Because Jaumann strains and stresses are defined as (Pai and Palazotto, 1994a)

$$B_{mn} = \frac{1}{2}\left(\frac{\partial\mathbf{u}}{\partial x_m}\cdot\mathbf{i}_n+\frac{\partial\mathbf{u}}{\partial x_n}\cdot\mathbf{i}_m\right) = \frac{\partial\mathbf{u}}{\partial x_m}\cdot\mathbf{i}_n = \frac{\partial\mathbf{u}}{\partial x_n}\cdot\mathbf{i}_m = B_{nm},$$

$$\frac{\mathbf{f}_1\cdot\mathbf{i}_1}{\mathrm{d}x_2\,\mathrm{d}x_3} = J_{11}, \quad \frac{\mathbf{f}_1\cdot\mathbf{i}_2}{\mathrm{d}x_2\,\mathrm{d}x_3}+\frac{\mathbf{f}_2\cdot\mathbf{i}_1}{\mathrm{d}x_1\,\mathrm{d}x_3} = 2J_{12} = 2J_{21} = J_{12}+J_{21}, \tag{56}$$

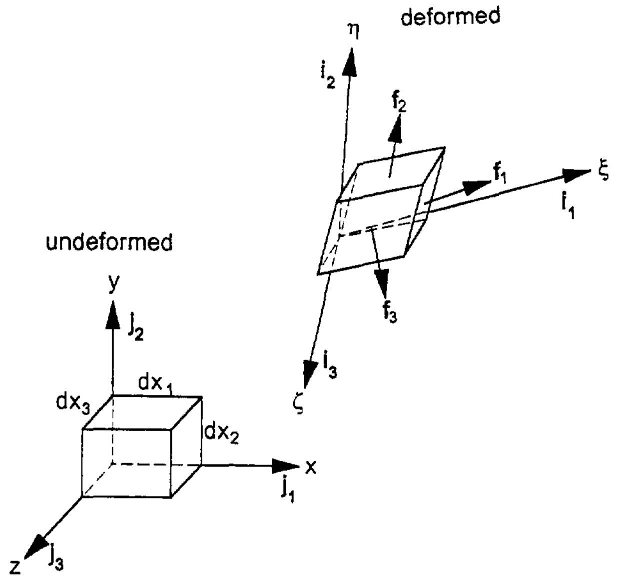

Fig. 2. The undeformed and deformed shapes of an infinitesimal element with the undeformed shape being a cube.

it follows from eqns (56) and (39a) that

$$\delta\Pi = \sum_{i=1}^{N}\int_A\int_{z_i}^{z_{i-1}}(J_{11}^{(i)}\delta B_{11}^{(i)}+J_{22}^{(i)}\delta B_{22}^{(i)}+J_{33}^{(i)}\delta B_{33}^{(i)}$$

$$+2J_{12}^{(i)}\delta B_{12}^{(i)}+2J_{23}^{(i)}\delta B_{23}^{(i)}+2J_{13}^{(i)}\delta B_{13}^{(i)})\,\mathrm{d}A\,\mathrm{d}z$$

$$= \sum_{i=1}^{N}\int_A\int_{z_i}^{z_{i+1}}\{\delta B^{(i)}\}^{\mathrm{T}}[\bar{Q}^{(i)}]\{B^{(i)}\}\,\mathrm{d}A\,\mathrm{d}z, \tag{57}$$

where A denotes the undeformed area of the reference surface. Substituting eqn (54a) into eqn (57) yields

$$\delta\Pi = \int_A\{\delta\psi\}^{\mathrm{T}}[\mathbf{\Phi}]\{\psi\}\,\mathrm{d}A, \tag{58}$$

where $[\mathbf{\Phi}]$ is a 12×12 symmetric matrix given by

$$[\mathbf{\Phi}] = \sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}[S^{(i)}]^{\mathrm{T}}[\bar{Q}^{(i)}][S^{(i)}]\,\mathrm{d}z. \tag{59}$$

Here, in order to simplify the analysis, we replace the deformed curvatures in eqns (54b) and (53a) with the undeformed curvatures. In practice, this approximation should not result in significant loss of accuracy because the curvatures in $[S_3^{(i)}]$ and $[S_4^{(i)}]$ in eqns (52a, b) are all multiplied by γ_4 and/or γ_5 and hence they are essentially nonlinear and small. Moreover, the curvatures cannot undergo large changes within the elastic range if it is a thick shell, and γ_4 and γ_5 are small if it is a thin shell.

It follows from eqns (46), (20)–(22), (24), (25), and (27)–(30) that the variation of the strain vector $\{\psi\}$ can be written as

$$\{\delta\psi\} = [\mathbf{\Psi}]\{\delta U\}, \tag{60}$$

where

$$\{U\} = \{u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, v, v_x, v_y, v_{xx}, v_{xy}, v_{yy}, w, w_x, w_y, w_{xx}, w_{xy}, w_{yy}, \gamma_4, \gamma_{4x}, \gamma_{4y}, \gamma_5, \gamma_{5x}, \gamma_{5y}\}^{\mathrm{T}} \tag{61a}$$

$$\Psi_{ij} = \frac{\partial \psi_i}{\partial U_j}. \tag{61b}$$

Similarly, one can put $\{\psi\}$ in the following form:

$$\{\psi\} = [\acute{\Psi}]\{U\}, \tag{62}$$

where

$$\acute{\Psi}_{ij} = \frac{\psi_i}{U_j}. \tag{63}$$

We note that $[\acute{\Psi}]$ may not be equal to $[\Psi]$. Substituting eqns (60) and (62) into eqn (58) yields

$$\delta\Pi = \int_A \{\delta U\}^{\mathrm{T}}[\Psi]^{\mathrm{T}}[\Phi][\acute{\Psi}]\{U\}\,\mathrm{d}A. \tag{64}$$

The way that the components of $\{U\}$ are approximated defines the type of a specific finite element. Using the finite-element discretization scheme, we discretize the displacements as

$$\{u, v, w, \gamma_4, \gamma_5\}^{\mathrm{T}} = [N]\{q^{[j]}\}, \tag{65}$$

where $\{q^{[j]}\}$ is the nodal displacement vector of the jth element and $[N]$ is a matrix of two-dimensional shape functions [see, e.g., Palazotto and Dennis (1992)]. Substituting eqn (65) into eqn (61a) yields

$$\{U\} = [D]\{q^{[j]}\} \tag{66}$$

$$[D] \equiv [\partial][N], \tag{67}$$

where $[\partial]$ is given in Appendix A.

Substituting eqn (66) into eqn (64) yields

$$\begin{aligned}\delta\Pi &= \sum_{j=1}^{N_e} \int_{A^{[j]}} \{\delta q^{[j]}\}^{\mathrm{T}}[D]^{\mathrm{T}}[\Psi]^{\mathrm{T}}[\Phi][\acute{\Psi}][D]\{q^{[j]}\}\,\mathrm{d}A \\ &= \sum_{j=1}^{N_e} \{\delta q^{[j]}\}^{\mathrm{T}}[K^{[j]}]\{q^{[j]}\} \\ &= \{\delta q\}^{\mathrm{T}}[K]\{q\},\end{aligned} \tag{68}$$

where

$$[K^{[j]}] \equiv \int_{A^{[j]}} [D]^{\mathrm{T}}[\Psi]^{\mathrm{T}}[\Phi][\acute{\Psi}][D]\,\mathrm{d}A. \tag{69}$$

N_e is the total number of elements, $A^{[j]}$ is the area of the jth element, $[K^{[j]}]$ is the stiffness matrix of the jth element, $[K]$ is the structural stiffness matrix, and $\{q\}$ is the structural displacement vector. We note that $[K^{[j]}]$ and hence $[K]$ may be asymmetric.

In analyses of nonlinear structures, because the structural stiffness matrix is a nonlinear function of displacements, the governing equations are usually solved by incremental/iterative methods. To derive linearized incremental equations, we let

$$\{q^{[j]}\} = \{q^0\} + \{\Delta q^{[j]}\}, \quad \{U\} = \{U^0\} + \{\Delta U\}, \tag{70}$$

where $\{q^0\}$ denotes the equilibrium solution and $\{\Delta q^{[j]}\}$ the increment displacement vector. Then, we obtain the first-order expansions of $\{\psi\}$ and $[\mathbf{\Psi}]$ as

$$\{\psi\} = \{\psi^0\} + [\mathbf{\Psi}^0]\{\Delta U\} \tag{71}$$

and

$$[\mathbf{\Psi}] = [\mathbf{\Psi}^0] + [\Xi], \tag{72}$$

where the entry Ξ_{ij} of $[\Xi]$ is given by

$$\Xi_{ij} = \frac{\partial^2 \psi_i}{\partial U_j \partial U_k} \Delta U_k. \tag{73}$$

Then, we use eqns (71) and (72) to expand $[K^{[j]}]\{q^{[j]}\}$ into a Taylor series and neglect higher-order terms to obtain

$$\begin{aligned} [K^{[j]}]\{q^{[j]}\} &= \int_{A^{[j]}} [D]^{\mathrm{T}}[\mathbf{\Psi}]^{\mathrm{T}}[\mathbf{\Phi}]\{\psi\}\,\mathrm{d}A \\ &= \int_{A^{[j]}} [[D]^{\mathrm{T}}[\mathbf{\Psi}^0]^{\mathrm{T}}[\mathbf{\Phi}]\{\psi^0\} + [D]^{\mathrm{T}}[\mathbf{\Psi}^0]^{\mathrm{T}}[\mathbf{\Phi}][\mathbf{\Psi}^0]\{\Delta U\} + [D]^{\mathrm{T}}[\Xi]^{\mathrm{T}}[\mathbf{\Phi}]\{\psi^0\}]\,\mathrm{d}A. \end{aligned} \tag{74}$$

Using eqn (73) and direct expansions, one can prove that

$$[\Xi]^{\mathrm{T}}[\mathbf{\Phi}]\{\psi^0\} = [\mathbf{\Upsilon}]\{\Delta U\}, \tag{75}$$

where $[\mathbf{\Upsilon}]$ is a symmetric matrix and its entry $\mathbf{\Upsilon}_{ij}$ is given by

$$\mathbf{\Upsilon}_{ij} = \mathbf{\Upsilon}_{ji} = \psi_m^0 \mathbf{\Phi}_{mn} \frac{\partial^2 \psi_n^0}{\partial U_i \partial U_j} = \psi_m^0 \mathbf{\Phi}_{mn} \frac{\partial \mathbf{\Psi}_{ni}^0}{\partial U_j}. \tag{76}$$

Hence, substituting eqns (75) and (66) into eqn (74) yields

$$[K^{[j]}]\{q^{[j]}\} = [\breve{K}^{[j]}]\{\Delta q^{[j]}\} + [K^{[j]}]\{q^{[j]}\}|_{\{q^{[j]}\} = \{q^0\}}, \tag{77}$$

where $[\breve{K}^{[j]}]$ is the so-called element tangent stiffness matrix and is given by

$$[\breve{K}^{[j]}] = \int_{A^{[j]}} [D]^{\mathrm{T}}([\mathbf{\Psi}^0]^{\mathrm{T}}[\mathbf{\Phi}][\mathbf{\Psi}^0] + [\mathbf{\Upsilon}])[D]\,\mathrm{d}A \tag{78}$$

and

$$[K^{[I]}]\{q^{[I]}\}|_{\{q^{[I]}\}=\{q^0\}} = \int_{A^{[I]}} [D]^{\mathrm{T}}[\boldsymbol{\Psi}^0]^{\mathrm{T}}[\boldsymbol{\Phi}]\{\psi^0\}\,\mathrm{d}A. \tag{79}$$

We note that $[\breve{K}^{[I]}]$ is a symmetric matrix.

4.2. *Kinetic energy*

The absolute displacement vector of an arbitrary point on the observed shell is

$$\mathbf{D} = u\mathbf{j}_1 + v\mathbf{j}_2 + w\mathbf{j}_3 + z\mathbf{i}_3 - z\mathbf{j}_3 + (\gamma_5 g_{15}^{(i)} + \gamma_4 g_{14}^{(i)})\mathbf{i}_1 + (\gamma_4 \mathbf{g}_{24}^{(i)} + \gamma_5 g_{25}^{(i)})\mathbf{i}_2 + (\alpha_3^{(i)} z + \beta_3^{(i)} z^2)\mathbf{i}_3. \tag{80}$$

Because translational displacements u, v, and w, rotations of the coordinate system $\xi\eta\zeta$, and shear rotations result in the main parts of the kinetic energy, inertias due to transverse normal stretching are negligible. Hence, we neglect normal deformations and take the variation of eqn (80) to obtain

$$\begin{aligned}\delta\mathbf{D} = {} & \mathbf{j}_1[\delta u + z\delta T_{31} + g_{15}^{(i)}\delta(\gamma_5 T_{11}) + g_{14}^{(i)}\delta(\gamma_4 T_{11}) + g_{24}^{(i)}\delta(\gamma_4 T_{21}) + g_{25}^{(i)}\delta(\gamma_5 T_{21})] \\ & + \mathbf{j}_2[\delta v + z\delta T_{32} + g_{15}^{(i)}\delta(\gamma_5 T_{12}) + g_{14}^{(i)}\delta(\gamma_4 T_{12}) + g_{24}^{(i)}\delta(\gamma_4 T_{22}) + g_{25}^{(i)}\delta(\gamma_5 T_{22})] \\ & + \mathbf{j}_3[\delta w + z\delta T_{33} + g_{15}^{(i)}\delta(\gamma_5 T_{13}) + g_{14}^{(i)}\delta(\gamma_4 T_{13}) + g_{24}^{(i)}\delta(\gamma_4 T_{23}) + g_{25}^{(i)}\delta(\gamma_5 T_{23})].\end{aligned} \tag{81}$$

Taking time derivatives of $\mathbf{D}$ yields

$$\begin{aligned}\ddot{\mathbf{D}} = {} & \mathbf{j}_1[\ddot{u} + z\ddot{T}_{31} + g_{15}^{(i)}(\gamma_5 T_{11})^{\cdot\cdot} + g_{14}^{(i)}(\gamma_4 T_{11})^{\cdot\cdot} + g_{24}^{(i)}(\gamma_4 T_{21})^{\cdot\cdot} + g_{25}^{(i)}(\gamma_5 T_{21})^{\cdot\cdot}] \\ & + \mathbf{j}_2[\ddot{v} + z\ddot{T}_{32} + g_{15}^{(i)}(\gamma_5 T_{12})^{\cdot\cdot} + g_{14}^{(i)}(\gamma_4 T_{12})^{\cdot\cdot} + g_{24}^{(i)}(\gamma_4 T_{22})^{\cdot\cdot} + g_{25}^{(i)}(\gamma_5 T_{22})^{\cdot\cdot}] \\ & + \mathbf{j}_3[\ddot{w} + z\ddot{T}_{33} + g_{15}^{(i)}(\gamma_5 T_{13})^{\cdot\cdot} + g_{14}^{(i)}(\gamma_4 T_{13})^{\cdot\cdot} + g_{24}^{(i)}(\gamma_4 T_{23})^{\cdot\cdot} + g_{25}^{(i)}(\gamma_5 T_{23})^{\cdot\cdot}].\end{aligned} \tag{82}$$

Using eqns (81) and (82), we obtain the variation of the kinetic energy K_{E} as

$$\begin{aligned}\delta K_{\mathrm{E}} &= -\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\int_A \rho^{(i)}\ddot{\mathbf{D}}\cdot\delta\mathbf{D}\,\mathrm{d}z\,\mathrm{d}A \\ &= -\int_A (\{\delta\hat{u}\}^{\mathrm{T}}[\hat{m}]\{\ddot{\hat{u}}\} + \{\delta\hat{v}\}^{\mathrm{T}}[\hat{m}]\{\ddot{\hat{v}}\} + \{\delta\hat{w}\}^{\mathrm{T}}[\hat{m}]\{\ddot{\hat{w}}\})\,\mathrm{d}A \\ &= -\int_A \{\delta\hat{\psi}\}^{\mathrm{T}}[m]\{\ddot{\hat{\psi}}\}\,\mathrm{d}A,\end{aligned} \tag{83}$$

where $\rho^{(i)}$ is the undeformed mass density of the ith layer,

$$\begin{aligned}\{\hat{u}\} &= \{u, T_{31}, \gamma_4 T_{11}, \gamma_5 T_{11}, \gamma_4 T_{21}, \gamma_5 T_{21}\}^{\mathrm{T}} \\ \{\hat{v}\} &= \{v, T_{32}, \gamma_4 T_{12}, \gamma_5 T_{12}, \gamma_4 T_{22}, \gamma_5 T_{22}\}^{\mathrm{T}} \\ \{\hat{w}\} &= \{w, T_{33}, \gamma_4 T_{13}, \gamma_5 T_{13}, \gamma_4 T_{23}, \gamma_5 T_{23}\}^{\mathrm{T}} \\ \{\hat{\psi}\} &= \{\{\hat{u}\}^{\mathrm{T}}, \{\hat{v}\}^{\mathrm{T}}, \{\hat{w}\}^{\mathrm{T}}\}^{\mathrm{T}} \\ [\hat{m}] &\equiv \sum_{i=1}^{N}\int_{z_i}^{z_{i+1}} \rho^{(i)}\{1, z, g_{14}^{(i)}, g_{15}^{(i)}, g_{24}^{(i)}, g_{25}^{(i)}\}^{\mathrm{T}}\{1, z, g_{14}^{(i)}, g_{15}^{(i)}, g_{24}^{(i)}, g_{25}^{(i)}\}\,\mathrm{d}z\end{aligned}$$

$$[\hat{m}] = \begin{bmatrix} I_0 & I_1 & I_5 & I_6 & I_7 & I_8 \\ I_1 & I_2 & I_{51} & I_{61} & I_{71} & I_{81} \\ I_5 & I_{51} & I_{55} & I_{56} & I_{57} & I_{58} \\ I_6 & I_{61} & I_{56} & I_{66} & I_{67} & I_{68} \\ I_7 & I_{71} & I_{57} & I_{67} & I_{77} & I_{78} \\ I_8 & I_{81} & I_{58} & I_{68} & I_{78} & I_{88} \end{bmatrix}$$

$$[m] \equiv \begin{bmatrix} [\hat{m}] & [0] & [0] \\ [0] & [\hat{m}] & [0] \\ [0] & [0] & [\hat{m}] \end{bmatrix} \tag{84}$$

$$\{I_0, I_1, I_2, I_5, I_6, I_7, I_8\} \equiv \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \rho^{(i)} \{1, z, z^2, g_{14}^{(i)}, g_{15}^{(i)}, g_{24}^{(i)}, g_{25}^{(i)}\} \, dz$$

$$\{I_{51}, I_{61}, I_{71}, I_{81}\} \equiv \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \rho^{(i)} \{z g_{14}^{(i)}, z g_{15}^{(i)}, z g_{24}^{(i)}, z g_{25}^{(i)}\} \, dz$$

$$\{I_{55}, I_{56}, I_{57}, I_{58}\} \equiv \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \rho^{(i)} \{g_{14}^{(i)} g_{14}^{(i)}, g_{14}^{(i)} g_{15}^{(i)}, g_{14}^{(i)} g_{24}^{(i)}, g_{14}^{(i)} g_{25}^{(i)}\} \, dz$$

$$\{I_{66}, I_{67}, I_{68}, I_{77}, I_{78}, I_{88}\} \equiv \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \rho^{(i)} \{g_{15}^{(i)} g_{15}^{(i)}, g_{15}^{(i)} g_{24}^{(i)}, g_{15}^{(i)} g_{25}^{(i)}, g_{24}^{(i)} g_{24}^{(i)}, g_{24}^{(i)} g_{25}^{(i)}, g_{25}^{(i)} g_{25}^{(i)}\} \, dz. \tag{85}$$

We note that $I_1 = 0$ if $\rho^{(i)} = \rho$ is constant and the middle surface is chosen as the reference surface.

It follows from eqns (84), (15), (10), (11), and (14) that the variation of $\{\hat{\psi}\}$ can be written as

$$\{\delta\hat{\psi}\} = [\hat{\mathbf{\Psi}}]\{\delta\hat{U}\}, \tag{86}$$

where

$$\{\hat{U}\} = \{u, u_x, u_y, v, v_x, v_y, w, w_x, w_y, \gamma_4, \gamma_5\}^{\mathrm{T}} \tag{87a}$$

$$\hat{\mathbf{\Psi}}_{ij} = \frac{\partial\hat{\psi}_i}{\partial\hat{U}_j}. \tag{87b}$$

Substituting eqn (86) into eqn (83) yields

$$\delta K_{\mathrm{E}} = -\int_A \{\delta\hat{U}\}^{\mathrm{T}} [\hat{\mathbf{\Psi}}]^{\mathrm{T}} [m] ([\hat{\mathbf{\Psi}}]\{\dot{\hat{U}}\}) \, dA. \tag{88}$$

Substituting eqn (65) into eqn (87a) yields

$$\{\hat{U}\} = [\hat{D}]\{q^{[j]}\}, \tag{89}$$

where

$$[\hat{D}] \equiv [\hat{c}][N] \tag{90}$$

and $[\hat{c}]$ is given in Appendix A.

Substituting eqn (89) into eqn (88) yields

$$\begin{aligned}\delta K_{\mathrm{E}} &= -\sum_{j=1}^{N_c}\int_{A^{[j]}}\{\delta q^{[j]}\}^{\mathrm{T}}[\hat{D}]^{\mathrm{T}}[\hat{\Psi}]^{\mathrm{T}}[m]([\hat{\Psi}][\hat{D}]\{\ddot{q}^{[j]}\}+[\dot{\hat{\Psi}}][\hat{D}]\{\dot{q}^{[j]}\})\,\mathrm{d}A \\ &= -\sum_{j=1}^{N_c}\{\delta q^{[j]}\}^{\mathrm{T}}([M^{[j]}]\{\ddot{q}^{[j]}\}+[\tilde{C}^{[j]}]\{\dot{q}^{[j]}\}) \\ &= -\{\delta q\}^{\mathrm{T}}([M]\{\ddot{q}\}+[\tilde{C}]\{\dot{q}\}),\end{aligned} \tag{91}$$

where

$$\begin{aligned}[M^{[j]}] &\equiv \int_{A^{[j]}}[\hat{D}]^{\mathrm{T}}[\hat{\Psi}]^{\mathrm{T}}[m][\hat{\Psi}][\hat{D}]\,\mathrm{d}A \\ [\tilde{C}^{[j]}] &\equiv \int_{A^{[j]}}[\hat{D}]^{\mathrm{T}}[\hat{\Psi}]^{\mathrm{T}}[m][\dot{\hat{\Psi}}][\hat{D}]\,\mathrm{d}A,\end{aligned} \tag{92}$$

$[\tilde{C}^{[j]}]$ is an artificial damping matrix due to inertias, and it may be asymmetric.

To derive linearized incremental equations, we let

$$\{q^{[j]}\} = \{q^0\}+\{\Delta q^{[j]}\}, \quad \{\hat{U}\} = \{\hat{U}^0\}+\{\Delta\hat{U}\}. \tag{93}$$

Then, we obtain the first-order expansions of $\{\hat{\psi}\}$ and $[\hat{\Psi}]$ as

$$\{\hat{\psi}\} = \{\hat{\psi}^0\}+[\hat{\Psi}^0]\{\Delta\hat{U}\} \tag{94}$$

and

$$[\hat{\Psi}] = [\hat{\Psi}^0]+[\hat{\Xi}], \tag{95}$$

where the entry $\hat{\Xi}_{ij}$ of $[\hat{\Xi}]$ is given by

$$\hat{\Xi}_{ij} = \frac{\partial^2\hat{\psi}_i}{\partial\hat{U}_j\,\partial\hat{U}_k}\Delta\hat{U}_k. \tag{96}$$

Then, we use eqns (94) and (95) to expand $[M^{[j]}]\{\ddot{q}^{[j]}\}+[\tilde{C}^{[j]}]\{\dot{q}^{[j]}\}$ into a Taylor series and neglect higher-order terms to obtain

$$\begin{aligned}&[M^{[j]}]\{\ddot{q}^{[j]}\}+[\tilde{C}^{[j]}]\{\dot{q}^{[j]}\} \\ &= \int_{A^{[j]}}[\hat{D}]^{\mathrm{T}}[\hat{\Psi}]^{\mathrm{T}}[m]\{\ddot{\hat{\psi}}\}\,\mathrm{d}A \\ &= \int_{A^{[j]}}([\hat{D}]^{\mathrm{T}}[\hat{\Psi}^0]^{\mathrm{T}}[m]\{\ddot{\hat{\psi}}^0\}+[\hat{D}]^{\mathrm{T}}[\hat{\Psi}^0]^{\mathrm{T}}[m][\hat{\Psi}^0]\{\Delta\ddot{\hat{U}}\}+2[\hat{D}]^{\mathrm{T}}[\hat{\Psi}^0]^{\mathrm{T}}[m][\dot{\hat{\Psi}}^0]\{\Delta\dot{\hat{U}}\} \\ &\quad +[\hat{D}]^{\mathrm{T}}[\hat{\Psi}^0]^{\mathrm{T}}[m][\ddot{\hat{\Psi}}^0]\{\Delta\hat{U}\}+[\hat{D}]^{\mathrm{T}}[\hat{\Xi}]^{\mathrm{T}}[m]\{\ddot{\hat{\psi}}^0\})\,\mathrm{d}A.\end{aligned} \tag{97}$$

Using eqn (96) and direct expansions, one can prove that

$$[\hat{\Xi}]^{\mathrm{T}}[m]\{\ddot{\hat{\psi}}^0\} = [\hat{\Upsilon}]\{\Delta U\}, \tag{98}$$

where $[\hat{\Upsilon}]$ is a symmetric matrix and its entry $\hat{\Upsilon}_{ij}$ is given by

$$\hat{\Upsilon}_{ij} = \hat{\Upsilon}_{ji} = \ddot{\hat{\psi}}^0_m m_{mn} \frac{\partial^2 \hat{\psi}^0_n}{\partial \hat{U}_i \partial \hat{U}_j} = \ddot{\hat{\psi}}^0_m m_{mn} \frac{\partial \hat{\Psi}^0_{ni}}{\partial \hat{U}_j}. \tag{99}$$

Hence, substituting eqns (98) and (89) into eqn (97) yields

$$[M^{[j]}]\{\ddot{q}^{[j]}\} + [\tilde{C}^{[j]}]\{\dot{q}^{[j]}\} = [\breve{M}^{[j]}]\{\Delta\ddot{q}^{[j]}\} + [M^{[j]}]\{\ddot{q}^{[j]}\}|_{\{q^{[j]}\}=\{q^0\}}$$
$$+ 2[\tilde{C}^{[j]}]|_{\{q^{[j]}\}=\{q^0\}}\{\Delta\dot{q}^{[j]}\} + [\tilde{K}^{[j]}]|_{\{q^{[j]}\}=\{q^0\}}\{\Delta q^{[j]}\}, \tag{100}$$

where

$$[\breve{M}^{[j]}] = \int_{A^{[j]}} [\hat{D}]^{\mathrm{T}}[\hat{\Psi}^0]^{\mathrm{T}}[m][\hat{\Psi}^0][\hat{D}]\,\mathrm{d}A \tag{101}$$

$$[M^{[j]}]\{\ddot{q}^{[j]}\}|_{\{q^{[j]}\}=\{q^0\}} = \int_{A^{[j]}} [\hat{D}]^{\mathrm{T}}[\hat{\Psi}^0]^{\mathrm{T}}[m]\{\ddot{\hat{\psi}}^0\}\,\mathrm{d}A \tag{102}$$

$$[\tilde{C}^{[j]}]_{\{q^{[j]}\}=\{q^0\}} = \int_{A^{[j]}} [\hat{D}]^{\mathrm{T}}[\hat{\Psi}^0]^{\mathrm{T}}[m][\dot{\hat{\Psi}}^0][\hat{D}]\,\mathrm{d}A \tag{103}$$

$$[\tilde{K}^{[j]}]_{\{q^{[j]}\}=\{q^0\}} = \int_{A^{[j]}} [\hat{D}]^{\mathrm{T}}([\hat{\Psi}^0]^{\mathrm{T}}[m][\ddot{\hat{\Psi}}^0] + [\hat{\Upsilon}])[\hat{D}]\,\mathrm{d}A. \tag{104}$$

Here, $[\breve{M}^{[j]}]$ is the so-called element tangent mass matrix, and it is a symmetric matrix. Moreover, $[\tilde{K}^{[j]}]$ is an artificial stiffness matrix due to inertias and it may be asymmetric.

4.3. *External loads*

It follows from eqn (65) that the variation of nonconservative energy due to external loads is

$$\begin{aligned}
\delta W_{\mathrm{nc}} &= \int_A \{\delta u, \delta v, \delta w, \delta\gamma_4, \delta\gamma_5\}\{r_1, r_2, r_3, 0, 0\}^{\mathrm{T}}\,\mathrm{d}A \\
&= \sum_{j=1}^{N_e} \int_{A^{[j]}} \{\delta q^{[j]}\}^{\mathrm{T}}[N]^{\mathrm{T}}\{r_1, r_2, r_3, 0, 0\}^{\mathrm{T}}\,\mathrm{d}A \\
&= \sum_{j=1}^{N_e} \{\delta q^{[j]}\}^{\mathrm{T}}\{R^{[j]}\} \\
&= \{\delta q\}^{\mathrm{T}}\{R\},
\end{aligned} \tag{105}$$

where r_1, r_2, and r_3 are distributed external loads along the directions of the axes x, y, and z, respectively. Here, we assume that r_1, r_2, and r_3 are not parametric loadings, that is, they are functions of x and y only and not functions of displacements u, v, and w. Moreover, $\{R\}$ is the structural nodal loading vector, and $\{R^{[j]}\}$ is the elemental nodal loading vector, which is given by

$$\{R^{[j]}\} \equiv \int_{A^{[j]}} [N]^{\mathrm{T}} \{r_1, r_2, r_3, 0, 0\}^{\mathrm{T}} \, \mathrm{d}A. \tag{106}$$

4.4. *Incremental equations*

An incremental/iterative method will be used to solve the derived fully nonlinear governing equations. Substituting eqns (68), (77), (91), (100), and (105) into eqn (55), we obtain the incremental equations of motion:

$$\sum_{j=1}^{N_e} ([\breve{M}^{[j]}]\{\Delta\ddot{q}^{[j]}\} + ([\breve{C}^{[j]}]+2[\tilde{C}^{[j]}])\{\Delta\dot{q}^{[j]}\} + ([\breve{K}^{[j]}]+[\tilde{K}^{[j]}])\{\Delta q^{[j]}\})$$

$$= \sum_{j=1}^{N_e} (\{R^{[j]}\} - [M^{[j]}]\{\ddot{q}^{[j]}\} - [C^{[j]}]\{\dot{q}^{[j]}\} - [K^{[j]}]\{q^{[j]}\})_{\{q^{[j]}\}=\{q^0\}}, \tag{107}$$

where the right-hand terms are shown in eqns (106), (102), and (79) and $[C^{[j]}]$ and $[\breve{C}^{[j]}]$ are assumed damping matrices due to internal and/or external dampings.

5. DISCUSSION

Substituting eqn (41) into eqn (32) yields

$$u_1^{(i)} = u_1^0 + z(\theta_2 - \theta_2^0) + \gamma_5 g_{15}^{(i)}(z) + \gamma_4 g_{14}^{(i)}(z)$$
$$u_2^{(i)} = u_2^0 - z(\theta_1 - \theta_1^0) + \gamma_4 g_{24}^{(i)}(z) + \gamma_5 g_{25}^{(i)}(z), \tag{108}$$

where $g_{15}^{(i)}(z)$ and $g_{24}^{(i)}(z)$ are called shear warping functions, and we call $g_{14}^{(i)}(z)$ and $g_{25}^{(i)}(z)$ shear coupling functions. For isotropic plates or one-layer orthotropic plates with an arbitrary ply angle, the shear functions are

$$g_{15}^{(i)} = g_{24}^{(i)} = z - \frac{4z^3}{3h^2}, \quad g_{14}^{(i)} = g_{25}^{(i)} = 0. \tag{109}$$

Thus, there is no coupling between the two transverse shear rotations γ_4 and γ_5. This is the so-called third-order shear-deformation theory (Bhimaraddi, 1984; Reddy, 1984). For all laminated plates except one-layer orthotropic plates and cross-ply laminated plates, $g_{14}^{(i)}$ and $g_{25}^{(i)}$ are nontrivial and hence γ_4 and γ_5 are linearly coupled. Pai *et al.* (1993) showed that skew-symmetric lamination results in even shear coupling functions whereas symmetric lamination results in odd shear coupling functions. For a general laminated anisotropic plate, the shear warping functions are not odd functions and the shear coupling functions are neither odd nor even functions. Pai and Nayfeh (1994a) showed that shear warping and coupling functions are initial curvature-dependent.

The piece-wise linear shear theory of Di Sciuva (1987) can account for the continuity of interlaminar shear stresses and is equivalent to setting, in eqn (43),

$$g_{14}^{(i)} = a_{14}^{(i)} + b_{14}^{(i)} z, \quad g_{15}^{(i)} = a_{15}^{(i)} + b_{15}^{(i)} z,$$
$$g_{24}^{(i)} = a_{24}^{(i)} + b_{24}^{(i)} z, \quad g_{25}^{(i)} = a_{25}^{(i)} + b_{25}^{(i)} z. \tag{110}$$

We note that this theory cannot accommodate the free shear stress conditions on the bonding surfaces and shear stresses are assumed to be constant within each layer.

There are three different ways of treating transverse normal stress and strain. The first one is to use $B_{33}^{(i)} = 0$, which is equivalent to using $u_3^{(i)} = u_3^0$ in eqn (32). The second one is to reduce the stiffness matrix $[\bar{Q}^{(i)}]$ in eqn (39a) by using the assumption $J_{33}^{(i)} = 0$, which is usually assumed for thin isotropic plates. The third one is to take the energy of transverse normal stress and strain into account. Because to solve a structural problem is to locate the minimum or the maximum point of its system energy, the third method, which is the one we propose here, is the most appropriate one, especially when highly anisotropic composite materials are concerned. The influence of transverse normal stress and strain on the structural stiffnesses can be seen from eqns (52a), (53a), (54), and (59) and the inclusion of $g_{3m}^{(i)}$ and $g_{4n}^{(i)}$ in eqn (53a).

In the presented shell model, the relative normal displacement (i.e. $u_3^{(i)}$) is assumed to be a displacement induced by the deformations of the reference surface [see eqns (32) and (50)] and the relative tangent displacements (i.e. $u_1^{(i)}$ and $u_2^{(i)}$) are assumed to be able to be described by the deformations of the reference surface and γ_4 and γ_5 [see eqns (32) and (41)], which imply that their motion frequency is the same as that of the motion of the reference surface. However, for moderately thick or even thin shells vibrating at very high frequencies, the motions of material points away from the reference surface can be independent of deformations of the reference surface, where a three-dimensional model is needed. Hence, the presented shell model is quasi-three-dimensional, not three-dimensional.

6. CLOSURE

Using new concepts of local displacements and orthogonal virtual rotations, Jaumann stress and strain measures, and an exact coordinate transformation, we developed a geometrically-exact shell theory which accounts for large rotations, large in-plane strains, arbitrary initial curvatures, transverse shear deformations, interlaminar normal stresses, continuity of interlaminar shear and peeling stresses, and three-dimensional stress effects. Moreover, elastic coupling effects between two transverse shear deformations are introduced and shear warping and coupling functions are obtained. Based on this comprehensive shell theory, a fully nonlinear displacement-based finite-element formulation is derived, which can be used to model any surface structure undergoing large rotations and strains. This finite-element model offers great flexibility in that plates and shells of arbitrary shapes consisting of arbitrarily oriented composite laminae with arbitrary stiffness properties can be modeled. The theory can be extended to include initial stresses and hygro-thermal-mechanical loading conditions.

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APPENDIX A

$$[\partial] \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \partial/\partial x & 0 & 0 & 0 & 0 \\ \partial/\partial y & 0 & 0 & 0 & 0 \\ \partial^2/\partial x^2 & 0 & 0 & 0 & 0 \\ \partial^2/\partial x\,\partial y & 0 & 0 & 0 & 0 \\ \partial^2/\partial y^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \partial/\partial x & 0 & 0 & 0 \\ 0 & \partial/\partial y & 0 & 0 & 0 \\ 0 & \partial^2/\partial x^2 & 0 & 0 & 0 \\ 0 & \partial^2/\partial x\,\partial y & 0 & 0 & 0 \\ 0 & \partial^2/\partial y^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \partial/\partial x & 0 & 0 \\ 0 & 0 & \partial/\partial y & 0 & 0 \\ 0 & 0 & \partial^2/\partial x^2 & 0 & 0 \\ 0 & 0 & \partial^2/\partial x\,\partial y & 0 & 0 \\ 0 & 0 & \partial^2/\partial y^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \partial/\partial x & 0 \\ 0 & 0 & 0 & \partial/\partial y & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \partial/\partial x \\ 0 & 0 & 0 & 0 & \partial/\partial y \end{bmatrix} \tag{A1}$$

$$[\hat{\partial}] \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \partial/\partial x & 0 & 0 & 0 & 0 \\ \partial/\partial y & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \partial/\partial x & 0 & 0 & 0 \\ 0 & \partial/\partial y & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \partial/\partial x & 0 & 0 \\ 0 & 0 & \partial/\partial y & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A2}$$